Arbitrage Pricing Theory Revisited: Structural Equation Models with Stochastic Constraints

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A Thesis Submitted in Partial Fulfilment
of the Requirements for the Degree of
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Abstract

Abstract of thesis entitled:

Arbitrage Pricing Theory Revisited: Structural Equation Models with Stochastic Constraints

Submitted by **Choy Man Wah Minnie**

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Arbitrage Pricing Theory (APT) is an important model for asset returns and it is widely used in financial market analysis. However, there exist some drawbacks of the traditional analysis. It has been proposed that some of these drawbacks can be addressed by using Structural Equation Model (SEM). In practical applications, some parameters are fixed to incorporate the prior knowledge into the analysis or to identify the model when SEM is used to analyse the APT model. However, analysts may be uncertain of the prior information, which leads to the difficulty of choosing appropriate values for the fixed parameters. The objective of the thesis is to address this problem by making use of stochastic constraints. In this thesis, we have developed a single stage procedure for APT analysis using SEM with stochastic constraints on model parameters. Stochastic constraints allow the fixed parameters to be differ from the prior information by a random error term, and therefore enable the interpretation of the analysis in a more realistic way. A nice feature of the proposed procedure is that it can be implemented by a widely available and easily accessible SEM package, namely the Mx. The proposed procedure has been used to analyse a set of data taken from 25 constituent stocks from the Hong Kong Hang Seng Index (HSI). Moreover, simulation studies have been conducted to investigate the performance of the proposed procedure.

摘要

在金融市場中，套利定價理論 (APT) 是一項被廣泛應用在資產回報的重要模型。然而，在傳統的分析中存有若干缺點。當中有建議指出有一些缺點可以應用結構方程模型來處理。在實際應用中，當我們利用結構方程模型來分析套利定價理論模型時，有些參數會被固定從而讓先驗信息結合在分析中或令模型得以確認。可是，分析員可能不確定先驗信息的確實性，從而令他們在選擇適當的數值予固定參數時遇到困難。本論文的目的是利用隨機約束來處理這個問題。在本論文中，我們引入隨機約束到利用單階段程序結構方程模型來分析套利定價理論之中。隨機約束容許固定參數和先驗信息的差數爲一個隨機誤差，因此能夠令此項分析的解說更寫實。當中我們所提出的程序的一個良好特點是我們可以利用 Mx 來實行它，而 Mx 是一個廣泛被應用和易於運用的結構方程模型程式。我們會利用所提出的程序來分析香港恆生指數中的二十五隻成份股。此外，我們會透過模擬研究來檢視這項程序的表現。

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Chapter 1

Introduction

Arbitrage Pricing Theory (APT) formulated by Ross (1976) constitutes an important asset pricing theory in modern finance (see Roll and Ross, 1980). However, there exist some drawbacks in the traditional two-step procedure analysis (see Roll and Ross, 1980) that hinder the theoretical development of APT and its uses in financial market. It has been proposed by Li (2004) that some of these drawbacks can be addressed by using Structural Equation Model (SEM). The analysis of APT using SEM will be reviewed in Chapter 2. When SEM is used to estimate APT model parameters in practice, it involves the use of prior information in the form of exact constraints on some unknown parameters or exact constraints are imposed on some parameters to identify the model. However, it may exist uncertainty of the prior information, which leads to the difficulty of choosing appropriate values for the fixed parameters.

In view of this, we proposed to address this problem by making use of stochastic constraints. Stochastic constraints are more flexible than exact constraints, and randomness are allowed to exist between the fixed parameters and their corresponding assigned values by an error term. Based on the analysis of stochastic constraints in SEM studied by Lee (1992), we developed a single stage procedure for APT analysis using SEM with stochastic constraints on model parameters. The detail of the analysis will be introduced in Chapter 3. A nice feature of the proposed procedure is that it can be implemented by a widely available and easily accessible SEM package, namely the Mx (Neale er al., 1999), and illustrative sample Mx input scripts are presented.

In Chapter 4, the results based on an empirical study are presented. The study illustrates the proposed procedure by using a set of data taken from 25 constituent stocks from the Hong Kong Hang Seng Index (HSI). Moreover, simulation studies have been conducted to investigate the performance of the proposed procedure, details of the results are summarized in Chapter 5. Finally, Chapter 6 gives a conclusion and discussion of the thesis.

Chapter 2

The Analysis of APT using SEM

2.1 The APT model

APT is a return generating process, it states that random returns of a set of p assets follow a K-factor generating model which is given by

$$r_i = \mu_i + b_{i1} f_1 + b_{i2} f_2 + \cdots + b_{iK} f_K + \epsilon_i \quad \text{for } i = 1, 2, \cdots, p \tag{2.1}$$

where $\mu_i = E(r_i)$; $f_1, f_2, \cdots, f_K$ are mean zero systematic risk factors common to the returns of all assets; $b_{i1}, b_{i2}, \cdots, b_{iK}$ are the factor loadings and ϵ_i is the error term of r_i. Besides, for the APT to be valid, it requires the number of assets considered (p) is much greater than the number of factors (K).

Consider an investor who currently holds a portfolio of p assets and is exploring to use the same amount of wealth to invest in an alterative portfolio that has greater expected return at the same level of risk. This alternative portfolio is called an arbitrage portfolio. Let $\mathbf{x} = (x_1, \cdots, x_p)^T$ be the changes in investment proportions of the p assets. Since no additional fund is added to the alterative portfolio, the proportions must satisfy

$$\mathbf{x}^T \mathbf{c} = 0 \tag{2.2}$$

where $\mathbf{c} = \lambda_0 \mathbf{1}$ is a $p \times 1$ vector with all elements equal to a constant λ_0. Therefore, by equation (2.1), the additional return gained from the arbitrage portfolio is:

$$\begin{aligned}
\mathbf{x}^T\mathbf{r} &= \sum_{i=1}^{p} x_i r_i \\
&= \sum_{i=1}^{p} x_i(\mu_i + b_{i1}f_1 + b_{i2}f_2 + \cdots + b_{iK}f_K + \epsilon_i) \\
&= \sum_{i=1}^{p} x_i\mu_i + \sum_{i=1}^{p} x_i b_{i1}f_1 + \sum_{i=1}^{p} x_i b_{i2}f_2 + \cdots + \sum_{i=1}^{p} x_i b_{iK}f_K + \sum_{i=1}^{p} x_i\epsilon_i \\
&= \mathbf{x}^T\boldsymbol{\mu} + \mathbf{x}^T\mathbf{b}_1 f_1 + \mathbf{x}^T\mathbf{b}_2 f_2 + \cdots + \mathbf{x}^T\mathbf{b}_K f_K + \mathbf{x}^T\boldsymbol{\epsilon} \qquad (2.3)
\end{aligned}$$

where $\mathbf{x} = (x_1, \cdots, x_p)^T$, $\mathbf{r} = (r_1, \cdots, r_p)^T$, $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_p)^T$, $\mathbf{b_j} = (b_{1j}, \cdots, b_{pj})^T$ for $j = 1, \cdots, K$ and $\boldsymbol{\epsilon} = (\epsilon_1, \cdots, \epsilon_p)^T$.

If the number of assets p is sufficient large, then the term $\mathbf{x}^T\epsilon$ converges to zero. Moreover, the arbitrage portfolio is formed without bearing additional risk, so the extra return shown by equation (2.3) should be unaffected by any systematic risk factor f_j, that means the terms $\mathbf{x}^T\mathbf{b}_j = 0$ for all $j = 1, \cdots, K$. As a result, the additional return of the arbitrage portfolio given by equation (2.3) becomes:

$$\mathbf{x}^T\mathbf{r} = \mathbf{x}^T\boldsymbol{\mu} \qquad (2.4)$$

In an equilibrium market, it is assumed that no arbitrage opportunities exist, so any arbitrage portfolio requires zero net investment and has zero risk should have zero expected payoff. Hence, $\mathbf{x}^T\mathbf{r} = \mathbf{x}^T\boldsymbol{\mu} = 0$. The foregoing can be restated as follows: for any $\mathbf{x}$ satisfying $\mathbf{x}^T\mathbf{c} = 0$, and $\mathbf{x}^T\mathbf{b}_j = 0$ for $j = 1, \cdots, K$ implies $\mathbf{x}^T\boldsymbol{\mu} = 0$. That is, any vector $\mathbf{x}$ orthogonal to $\mathbf{c}$ and $\mathbf{b}_j$ for $j = 1, \cdots, K$ is also orthogonal to $\boldsymbol{\mu}$. From a standard result of linear algebra, it implies that $\boldsymbol{\mu}$ is a linear combination of $\mathbf{c}$ and $\mathbf{b}_1, \cdots, \mathbf{b}_p$. Thus, there exists constants $\lambda_0, \lambda_1, \cdots, \lambda_K$ such that

$$\mu_i = \lambda_0 + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \cdots + \lambda_K b_{iK} \quad \text{for } i = 1, 2, \cdots, p \qquad (2.5)$$

Consider a risk free asset $i = f$ which is insensitive to any systematic factor, that is $b_{fj} = 0$ for all j. From equation (2.1), its return will be $r_f = \mu_f$ and from equation (2.5), its expected return will be $\mu_f = \lambda_0$. By combining the two equations, it shows that $\lambda_0 = r_f$, and the pricing relationship of the APT is given by:

$$\mu_i = r_f + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \cdots + \lambda_K b_{iK} \quad \text{for } i = 1, 2, \cdots, p \tag{2.6}$$

It states that the excess expected return $\mu_i - r_f$ is a linear combination of the systematic factor loading $\{b_{ij}\}$ and with coefficient λ_j.

λ_j is called the factor risk premia and it can be interpreted as the excess return or market risk premium on assets with only systematic factor j. Consider a portfolio with expected return $\mu^{(k)}$ and has only a unit systematic risk on the k^{th} factor; that is, $b_{ik} = 1$ and all other $b_{ij} = 0$ for $j \neq k$. This leads to

$$\begin{aligned} \mu^{(k)} &= r_f + \lambda_k \quad \text{or} \\ \lambda_k &= \mu^{(k)} - r_f \end{aligned}$$

Therefore, the pricing equation in (2.6) can also be written as:

$$\begin{aligned} \mu_i &= r_f + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \cdots + \lambda_K b_{iK} \\ &= r_f + (\mu^{(1)} - r_f) b_{i1} + (\mu^{(2)} - r_f) b_{i2} + \cdots + (\mu^{(K)} - r_f) b_{iK} \end{aligned} \tag{2.7}$$

2.2 The structural equation model approach

Combining the Factor Model (2.1) and the Pricing Equation (2.6), it gives

$$\begin{aligned} r_i &= \mu_i + b_{i1} f_1 + b_{i2} f_2 + \cdots + b_{iK} f_K + \epsilon_i \\ &= r_f + \lambda_1 b_{i1} + \lambda_2 b_{i2} + \cdots + \lambda_K b_{iK} + b_{i1} f_1 + b_{i2} f_2 + \cdots + b_{iK} f_K + \epsilon_i \\ &= r_f + b_{i1}(\lambda_1 + f_1) + b_{i2}(\lambda_2 + f_2) + \cdots + b_{iK}(\lambda_K + f_K) + \epsilon_i \\ &= r_f + b_{i1} f_1' + b_{i2} f_2' + \cdots + b_{iK} f_K' + \epsilon_i \quad \text{for all } i \end{aligned} \tag{2.8}$$

where $f'_j = \lambda_j + f_j$. The mean of f'_j is λ_j and its variance is the same as that of f_j. In matrix form, the APT model is given by the following equation:

$$\begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_p \end{pmatrix} = \begin{pmatrix} r_f \\ r_f \\ \vdots \\ r_f \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1K} \\ b_{21} & b_{22} & \cdots & b_{2K} \\ \vdots & \vdots & \cdots & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pK} \end{pmatrix} \begin{pmatrix} f'_1 \\ f'_2 \\ \vdots \\ f'_K \end{pmatrix} + \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_p \end{pmatrix} \tag{2.9}$$

where p is the number of observed variables $(r_i, i = 1, \cdots, p)$; $f'_i = \lambda_i + f_i$ for $i = 1, \cdots, K$ with $\mathrm{E}(f'_i) = \lambda_i$, $\mathrm{E}(f_i) = 0$ and K is the number of latent factors. In order to analyse the model using the SEM technique, it is observed that the APT model in equation (2.9) can be written as a measurement equation with mean structure of a confirmatory factor analysis model (see, Joreskog & Sorbom, 1988). The measurement equation is given by:

$$\mathbf{r} = r_f\mathbf{1} + \mathbf{B}\boldsymbol{\xi} + \boldsymbol{\epsilon}$$

where $\mathbf{r}$ is a $p \times 1$ vector of the p assets; $\mathbf{1}$ is a $p \times 1$ vector with all elements equal to 1; $\mathbf{B}$ is the $p \times K$ factor loadings matrix; $\boldsymbol{\xi} = (f'_1, \cdots, f'_K)^T$ is a $K \times 1$ vector of latent variables and $\boldsymbol{\epsilon}$ is a $p \times 1$ vector of random errors, with $\mathrm{E}(\boldsymbol{\epsilon}) = 0$. The expected value of $\mathbf{r}$ is given by:

$$\begin{aligned} \mathrm{E}(\mathbf{r}) &= \mathrm{E}(r_f\mathbf{1}) + \mathbf{B}\mathrm{E}(\boldsymbol{\xi}) + \mathrm{E}(\boldsymbol{\epsilon}) \\ &= r_f\mathbf{1} + \mathbf{B}\boldsymbol{\Lambda} \end{aligned} \tag{2.10}$$

where $\boldsymbol{\Lambda} = \mathrm{E}(\boldsymbol{\xi}) = (\lambda_1, \cdots, \lambda_K)^T$ is the mean vector of the latent factors. In addition, the Covariance matrix of $\mathbf{r}$ is:

$$\begin{aligned} \mathrm{Cov}(\mathbf{r}) &= \mathbf{B}\mathrm{Cov}(\boldsymbol{\xi})\mathbf{B}' + \mathrm{Cov}(\boldsymbol{\epsilon}) \\ &= \mathbf{B}\boldsymbol{\Phi}\mathbf{B}' + \boldsymbol{\Theta}_{\epsilon} \end{aligned} \tag{2.11}$$

where $\boldsymbol{\Phi} = \mathrm{Cov}(\boldsymbol{\xi})$ is the $K \times K$ covariance matrix of latent factors; $\boldsymbol{\Theta}_{\boldsymbol{\epsilon}} = \mathrm{Cov}(\boldsymbol{\epsilon})$ is the $p \times p$ covariance matrix of error terms. It is a common practice to consider $\boldsymbol{\Theta}_{\boldsymbol{\epsilon}}$ as a diagonal matrix because the error terms are usually assumed to be independent.

In general, it can be seen from equations (2.10) and (2.11) that all unknown parameters in the model are involved in the mean vector $\mathrm{E}(\mathbf{r})$ and the covariance matrix $\mathrm{Cov}(\mathbf{r})$. Let $\boldsymbol{\theta}$ be the vector of all unknown parameters, it consists of r_f and elements in $\boldsymbol{\Lambda}$, $\mathbf{B}$, $\boldsymbol{\Phi}$ and $\boldsymbol{\Theta}_{\epsilon}$. In practice, some of them are fixed at pre-assigned value and known as fixed parameters; some are constrained parameters which are unknown but equal to one or more other parameters; and the others are free unknown parameters. For a SEM with mean structure, one factor loading for each latent factor and the intercept term are usually fixed at some specific value for the purpose of identification. Furthermore, we can also incorporate the prior information to the model by imposing the exact constraints to some parameters. Give a set of observations for $\mathbf{r}$, say $r_1, \cdots, r_n$, an estimate $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ can be obtained using the SEM approach.

Chapter 3

Incorporating stochastic constraints into the SEM analysis of APT

As mentioned in the previous chapter, some exact constraints are imposed on the parameters in order to identify the SEM. For example, we need to fixed r_f (elements of intercept vector) and the factor loadings of the reference variables, which are so-called the reference factor loadings, at specific values. In practical applications, these values may be difficult to determine accurately. To deal with this problem, we will make use of stochastic constraints. In other words, instead of fixing the parameters at an assigned value, we identify the model by imposing stochastic constraints on these parameters which will be estimated in the analysis.

3.1 Introduction

Firstly, we consider the stochastic constraint that is defined by:

$$\mathbf{u} = \mathbf{g}(\boldsymbol{\theta}) + \boldsymbol{\epsilon}_{\boldsymbol{u}} \tag{3.1}$$

where $\mathbf{u}$ is a $m \times 1$ vector to specify the prior information; $\mathbf{g}(\boldsymbol{\theta})$ is a $m \times 1$ vector of differentiable functions of the vector $\boldsymbol{\theta}$; $\boldsymbol{\epsilon_u}$ is a $m \times 1$ vector of random error measurement with distribution $N(0,\boldsymbol{\Gamma})$ and m is the number of constraints. Thus

$$\mathbf{u} - \mathbf{g}(\boldsymbol{\theta}) = \boldsymbol{\epsilon_u}$$

$$\mathbf{u} - \mathbf{g}(\boldsymbol{\theta}) \sim N(0, \boldsymbol{\Gamma})$$

$$\Pr(\mathbf{u} \mid \boldsymbol{\theta}, \boldsymbol{\Gamma}) \propto |\boldsymbol{\Gamma}|^{-\frac{1}{2}}\exp\{-[\mathbf{u} - \mathbf{g}(\boldsymbol{\theta})]'\boldsymbol{\Gamma}^{-1}[\mathbf{u} - \mathbf{g}(\boldsymbol{\theta})]/2\} \tag{3.2}$$

From equation (3.1), it is clear that when the covariance matrix $\boldsymbol{\Gamma}$ of $\boldsymbol{\epsilon_u}$ tends to zero, the stochastic constraints reduce to exact constraints. Hence, exact constraints can be considered as a special case of stochastic constraints.

3.2 Bayesian analysis of stochastic constraints

Consider a sample $\{\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_n\}$ of $\mathbf{r}$ with size n from a p-dimensional multivariate normal distribution with a $p \times 1$ mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}(\boldsymbol{\theta})$ where $\boldsymbol{\Sigma}(\boldsymbol{\theta})$ is a $p \times p$ matrix whose elements are differentiable functions of the parameter vector $\boldsymbol{\theta}$. From the n observations of $\mathbf{r_i}$, we get the $\bar{\mathbf{r}}$ and $\mathbf{S}$ which represent the sample mean vector and the maximum likelihood estimator of the covariance matrix of $\mathbf{r}$ respectively. These give the likelihood function in the form: (see, Johnson & Wichern, 2002)

$$\begin{aligned}
&\Pr(\mathbf{S}, \bar{\mathbf{r}} \mid \boldsymbol{\theta}) \\
&\propto \ |\boldsymbol{\Sigma}|^{-\frac{n}{2}}\exp\{-\frac{1}{2}\mathrm{tr}[\boldsymbol{\Sigma}^{-1}(\sum_{j=1}^{n}(\mathbf{r}_j - \bar{\mathbf{r}})(\mathbf{r}_j - \bar{\mathbf{r}})' + n(\bar{\mathbf{r}} - \boldsymbol{\mu})(\bar{\mathbf{r}} - \boldsymbol{\mu})')]\} \\
&= \ |\boldsymbol{\Sigma}|^{-\frac{n}{2}}\exp\{-\frac{1}{2}\mathrm{tr}[\boldsymbol{\Sigma}^{-1}(\sum_{j=1}^{n}(\mathbf{r}_j - \bar{\mathbf{r}})(\mathbf{r}_j - \bar{\mathbf{r}})')] - \frac{n}{2}(\bar{\mathbf{r}} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{r}} - \boldsymbol{\mu})\} \\
&= \ |\boldsymbol{\Sigma}|^{-\frac{n}{2}}\exp[-\frac{n}{2}\mathrm{tr}(\mathbf{S}\boldsymbol{\Sigma}^{-1}) - \frac{n}{2}(\bar{\mathbf{r}} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{r}} - \boldsymbol{\mu}))]
\end{aligned} \tag{3.3}$$

The joint posterior density of $\boldsymbol{\theta}$ and $\boldsymbol{\Gamma}$ is given by:

$$
\begin{aligned}
\Pr(\boldsymbol{\theta}, \boldsymbol{\Gamma} \mid \mathbf{S}, \bar{\mathbf{r}}, \mathbf{u}) &= \frac{\Pr(\boldsymbol{\theta}, \boldsymbol{\Gamma}, \mathbf{S}, \bar{\mathbf{r}}, \mathbf{u})}{\Pr(\mathbf{S}, \bar{\mathbf{r}}, \mathbf{u})} \\
&\propto \Pr(\boldsymbol{\theta}, \boldsymbol{\Gamma}, \mathbf{S}, \bar{\mathbf{r}}, \mathbf{u}) \\
&= \Pr(\mathbf{S}, \bar{\mathbf{r}}, \mathbf{u} \mid \boldsymbol{\theta}, \boldsymbol{\Gamma}) \Pr(\boldsymbol{\theta}, \boldsymbol{\Gamma}) \\
&= \Pr(\mathbf{S}, \bar{\mathbf{r}} \mid \boldsymbol{\theta}, \boldsymbol{\Gamma}) \Pr(\mathbf{u} \mid \boldsymbol{\theta}, \boldsymbol{\Gamma}) \Pr(\boldsymbol{\theta}) Pr(\boldsymbol{\Gamma}) \\
&\propto \Pr(\mathbf{S}, \bar{\mathbf{r}} \mid \boldsymbol{\theta}) \Pr(\mathbf{u} \mid \boldsymbol{\theta}, \boldsymbol{\Gamma}) \Pr(\boldsymbol{\Gamma}) \qquad (3.4)
\end{aligned}
$$

where the probability density function of $\boldsymbol{\theta}$ is assumed to be proportional to a constant when there is no prior knowledge on $\boldsymbol{\theta}$. By substituting some prior information represented by equations (3.2) and (3.3) into (3.4), the joint posterior density becomes:

$$
\begin{aligned}
\Pr(\boldsymbol{\theta}, \boldsymbol{\Gamma} \mid \mathbf{S}, \bar{\mathbf{r}}, \mathbf{u}) \propto\ & |\boldsymbol{\Sigma}|^{-\frac{n}{2}} \exp[-\frac{n}{2}\mathrm{tr}(\mathbf{S}\boldsymbol{\Sigma}^{-1}) - \frac{n}{2}(\bar{\mathbf{r}} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{r}} - \boldsymbol{\mu})] \\
& \times |\boldsymbol{\Gamma}|^{-\frac{1}{2}} \exp\{\frac{-[\mathbf{u} - \mathbf{g}(\boldsymbol{\theta})]'\boldsymbol{\Gamma}^{-1}[\mathbf{u} - \mathbf{g}(\boldsymbol{\theta})]}{2}\} \Pr(\boldsymbol{\Gamma}) \qquad (3.5)
\end{aligned}
$$

3.3 Three types of structures for $\boldsymbol{\Gamma}$

In this section, three different cases of stochastic constraints proposed by Lee (1992) that reflect three different structures of $\boldsymbol{\Gamma}$ are considered. In the first case, $\boldsymbol{\Gamma}$ is a diagonal matrix with equal diagonal elements. In the second case, $\boldsymbol{\Gamma}$ is a diagonal matrix with different diagonal elements. In the third case, $\boldsymbol{\Gamma}$ is a general positive definite matrix. The three cases correspond to different relationships of the error terms in the stochastic constraints and are sufficient for most of the practical applications. More details are given in the following section.

3.3.1 Case 1: $\boldsymbol{\Gamma} = \sigma^2 \mathbf{I}_{m \times m}$

In this case the random errors of the stochastic constraints are assumed to be independent of each other and with the same variances σ^2. Therefore, equation

(3.2) becomes:

$$\Pr(\mathbf{u} \mid \boldsymbol{\theta}, \boldsymbol{\Gamma}) \propto (\sigma^2)^{-\frac{m}{2}} \exp\{-[\frac{\sum_{j=1}^{m}(u_j - g_j)^2}{2\sigma^2}]\} \tag{3.6}$$

In addition, by using conjugate family, we assume the prior distribution of σ^2 is inverse χ^2 (see Lee, 1992). Hence, for given prior constants ν and β, $\frac{\nu\beta}{\sigma^2}$ is distributed as χ^2_ν.

$$\Pr(\sigma^2 \mid \nu, \beta) \propto (\sigma^2)^{-\frac{\nu+2}{2}} \exp(\frac{-\nu\beta}{2\sigma^2}) \tag{3.7}$$

From Lee (1992), it is known that the prior distribution has a single mode at the value

$$\sigma^2_{\max} = \beta(\frac{\nu}{\nu+2})$$

This gives the relationship between σ^2, β and ν, the value of σ^2 gets closer to β when ν becomes larger. In general, if ν is sufficiently large and β is sufficiently small, then $\sigma^2_{\max}$ will close to zero. That means the stochastic constraints become exact constraints. After substituting equations (3.6) and (3.7) to (3.5), we have

$$\begin{aligned} \Pr(\boldsymbol{\theta}, \boldsymbol{\Gamma} \mid \mathbf{S}, \bar{\mathbf{r}}, \mathbf{u}) &\propto |\boldsymbol{\Sigma}|^{-\frac{n}{2}} \exp[-\frac{n}{2}\mathrm{tr}(\mathbf{S}\boldsymbol{\Sigma}^{-1}) - \frac{n}{2}(\bar{\mathbf{r}} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{r}} - \boldsymbol{\mu})] \\ &\quad \times (\sigma^2)^{-\frac{\nu+m+2}{2}} \exp\{-[\frac{\sum_{j=1}^{m}(u_j - g_j)^2 + \nu\beta}{2\sigma^2}]\} \end{aligned}$$

and by the result in Lee (1992)

$$\int_0^\infty (\sigma^2)^{-\frac{\nu+m+2}{2}} \exp\{-[\frac{\sum_{j=1}^{m}(u_j - g_j)^2 + \nu\beta}{2\sigma^2}]\}\, \mathrm{d}\sigma^2 \propto [\sum_{j=1}^{m}(u_j - g_j)^2 + \nu\beta]^{-(\frac{\nu+m}{2})}$$

The nuisance parameter σ^2 can be integrated out and the posterior density of $\boldsymbol{\theta}$ can be expressed as:

$$\begin{aligned} \Pr(\boldsymbol{\theta} \mid \mathbf{S}, \bar{\mathbf{r}}, \mathbf{u}) &\propto |\boldsymbol{\Sigma}|^{-\frac{n}{2}} \exp[-\frac{n}{2}\mathrm{tr}(\mathbf{S}\boldsymbol{\Sigma}^{-1}) - \frac{n}{2}(\bar{\mathbf{r}} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{r}} - \boldsymbol{\mu})] \\ &\quad \times [\sum_{j=1}^{m}(u_j - g_j)^2 + \nu\beta]^{-(\frac{\nu+m}{2})} \end{aligned} \tag{3.8}$$

After taking logarithm of equation (3.8), we get the log-likelihood function:

$$
\begin{aligned}
\log\Pr(\boldsymbol{\theta} \mid \mathbf{S}, \bar{\mathbf{r}}, \mathbf{u}) &\propto -(\frac{n}{2})\log|\boldsymbol{\Sigma}| - \frac{n}{2}\mathrm{tr}(\mathbf{S}\boldsymbol{\Sigma}^{-1}) - \frac{n}{2}(\bar{\mathbf{r}} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{r}} - \boldsymbol{\mu}) \\
&\quad -(\frac{\nu + m}{2})\log[\sum_{j=1}^{m}(u_j - g_j)^2 + \nu\beta] \\
&\propto -n\log|\boldsymbol{\Sigma}| - (n)\mathrm{tr}(\mathbf{S}\boldsymbol{\Sigma}^{-1}) - n(\bar{\mathbf{r}} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{r}} - \boldsymbol{\mu}) \\
&\quad -(\nu + m)\log[\sum_{j=1}^{m}(u_j - g_j)^2 + \nu\beta]
\end{aligned}
$$

Since $\mathbf{S}$ and p do not involve the unknown parameters, hence we can modify the log-likelihood into the following form:

$$
\begin{aligned}
\log\Pr(\boldsymbol{\theta} \mid \mathbf{S}, \bar{\mathbf{r}}, \mathbf{u}) &\propto -n[\log|\boldsymbol{\Sigma}| - \log|\mathbf{S}| + \mathrm{tr}(\mathbf{S}\boldsymbol{\Sigma}^{-1}) - p] \\
&\quad -n(\bar{\mathbf{r}} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{r}} - \boldsymbol{\mu}) - (\nu + m)\log[\sum_{j=1}^{m}(u_j - g_j)^2 + \nu\beta]
\end{aligned}
$$

Such modification will facilitate the optimisation be implemented in widely available software. Now, the estimate of $\boldsymbol{\theta}$ can be obtained by maximising $\log\Pr(\boldsymbol{\theta} \mid \mathbf{S}, \bar{\mathbf{r}}, \mathbf{u})$ that is the same as minimising $-\log\Pr(\boldsymbol{\theta} \mid \mathbf{S}, \bar{\mathbf{r}}, \mathbf{u})$. In conclusion, the Bayesian estimate $\hat{\boldsymbol{\theta}}_{\mathbf{1}}$ can be obtained by minimising the function:

$$F_1(\boldsymbol{\theta}) = F(\boldsymbol{\theta}) + B_1(\boldsymbol{\theta})$$

where

$$F(\boldsymbol{\theta}) = n[\log|\Sigma| - \log|\mathbf{S}| + \mathrm{tr}(\mathbf{S}\boldsymbol{\Sigma}^{-1}) - p] + n(\bar{\mathbf{r}} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{r}} - \boldsymbol{\mu}) \qquad (3.9)$$

and

$$B_1(\boldsymbol{\theta}) = (\nu + m)\log[\sum_{j=1}^{m}(u_j - g_j)^2 + \nu\beta]$$

3.3.2 Case 2: Γ is a diagonal matrix with diagonal elements σ_j^2 for $j = 1, \cdots, m$.

This implies that the error components in ϵ of equation (3.2) are independent, with different variances σ_j^2. By using conjugate family to specify the prior distribution of σ_j^2 with the given prior constants ν_j and β_j, we assume $\frac{\nu_j \beta_j}{\sigma_j^2}$ independently distributed as $\chi^2_{\nu_j}$. Hence, we have

$$\Pr(\mathbf{u} \mid \boldsymbol{\theta}, \boldsymbol{\Gamma}) \propto \prod_{j=1}^{m} \{(\sigma_j^2)\exp[-\frac{(u_j - g_j)^2}{2\sigma_j^2}]\} \tag{3.10}$$

$$\Pr(\sigma_j^2 \mid \nu_j, \beta_j) \propto (\sigma_j^2)^{-\frac{\nu_j+2}{2}} \exp(\frac{-\nu_j \beta_j}{2\sigma_j^2}) \tag{3.11}$$

Similar to that in Case 1 (see also Lee, 1992), the prior distribution of σ_j^2 has a single mode at value

$$\sigma^2_{j,\ \max} = \beta_j(\frac{\nu_j}{\nu_j + 2})$$

Therefore, similar relationship between $\sigma^2_{j,\ \max}$, β_j and ν_j is obtained. Moreover, by substituting equations (3.3), (3.10) and (3.11) to (3.4), it can be shown that

$$\begin{aligned} &\Pr(\boldsymbol{\theta}, \sigma_1^2, \sigma_2^2, \cdots, \sigma_m^2 \mid \mathbf{S}, \bar{\mathbf{r}}, \mathbf{u}) \\ &\propto\ |\boldsymbol{\Sigma}|^{-\frac{n}{2}} \exp[-\frac{n}{2}\mathrm{tr}(\mathbf{S}\boldsymbol{\Sigma}^{-1}) - \frac{n}{2}(\bar{\mathbf{r}} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{r}} - \boldsymbol{\mu})] \\ &\quad \times \prod_{j=1}^{m} (\sigma_j^2)^{-\frac{\nu_j+3}{2}} \exp\{-[\frac{(u_j - g_j)^2 + \nu_j\beta_j}{2\sigma_j^2}]\} \end{aligned}$$

From Lee (1992), the nuisance parameters σ_j^2, $j = 1, \cdots, m$, can be eliminated by the following integral.

$$\int_0^{\infty} (\sigma_j^2)^{-\frac{\nu_j+3}{2}} \exp\{-[\frac{(u_j - g_j)^2 + \nu_j\beta_j}{2\sigma_j^2}]\}\ d\sigma_j^2 \propto [(u_j - g_j)^2 + \nu_j\beta_j]^{-(\frac{\nu_j+1}{2})}$$

Therefore the posterior density of $\boldsymbol{\theta}$ becomes:

$$\Pr(\boldsymbol{\theta} \mid \mathbf{S}, \bar{\mathbf{r}}, \mathbf{u})\ \propto\ |\boldsymbol{\Sigma}|^{-\frac{n}{2}} \exp[-\frac{n}{2}\mathrm{tr}(\mathbf{S}\boldsymbol{\Sigma}^{-1}) - \frac{n}{2}(\bar{\mathbf{r}} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{r}} - \boldsymbol{\mu})]$$

$$\times \prod_{j=1}^{m}[(u_j - g_j)^2 + \nu_j\beta_j]^{-(\frac{\nu_j+1}{2})}$$

and the log-likelihood function is given by:

$$\begin{aligned}
\text{logPr}(\boldsymbol{\theta} \mid \mathbf{S}, \bar{\mathbf{r}}, \mathbf{u}) &\propto -(\frac{n}{2})\text{log}|\boldsymbol{\Sigma}| - \frac{n}{2}\text{tr}(\mathbf{S}\boldsymbol{\Sigma}^{-1}) - \frac{n}{2}(\bar{\mathbf{r}} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{r}} - \boldsymbol{\mu}) \\
&\quad - \sum_{j=1}^{m}(\frac{\nu_j + 1}{2})\text{log}[(u_j - g_j)^2 + \nu_j\beta_j] \\
&\propto -n[\text{log}|\boldsymbol{\Sigma}| - \text{log}|\mathbf{S}| + \text{tr}(\mathbf{S}\boldsymbol{\Sigma}^{-1}) - p] - n(\bar{\mathbf{r}} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{r}} - \boldsymbol{\mu}) \\
&\quad - \sum_{j=1}^{m}(\nu_j + 1)\text{log}[(u_j - g_j)^2 + \nu_j\beta_j]
\end{aligned}$$

As a result, the Bayesian estimate $\hat{\boldsymbol{\theta}}_2$ of $\boldsymbol{\theta}$ can be obtained by minimising the function:

$$F_2(\boldsymbol{\theta}) = F(\boldsymbol{\theta}) + B_2(\boldsymbol{\theta})$$

where F($\boldsymbol{\theta}$) is the same as that in equation (3.9) and is given by

$$F(\boldsymbol{\theta}) = n[\text{log}|\boldsymbol{\Sigma}| - \text{log}|\mathbf{S}| + \text{tr}(\mathbf{S}\boldsymbol{\Sigma}^{-1}) - p] + n[(\bar{\mathbf{r}} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{r}} - \boldsymbol{\mu})]$$

and

$$B_2(\boldsymbol{\theta}) = \sum_{j=1}^{m}(\nu_j + 1)\text{log}[(u_j - g_j)^2 + \nu_j\beta_j]$$

3.3.3 Case 3: $\boldsymbol{\Gamma}$ is a general positive definite matrix.

By using the conjugate distribution for $\boldsymbol{\Gamma}$, we assume $\boldsymbol{\Gamma}$ has independent inverted Wishart distribution with known positive definite matrix $\mathbf{R}$ and known degrees of freedom ρ. Thus, we get

$$\text{Pr}(\mathbf{u} \mid \boldsymbol{\theta}, \boldsymbol{\Gamma}) \propto |\boldsymbol{\Gamma}|^{-\frac{1}{2}}\text{exp}\{-[\text{tr}(\mathbf{A})\boldsymbol{\Gamma}^{-1}/2]\} \qquad (3.12)$$

where $\mathbf{A} = [\mathbf{u} - \mathbf{g}(\boldsymbol{\theta})][\mathbf{u} - \mathbf{g}(\boldsymbol{\theta})]'$ and

$$\text{Pr}(\boldsymbol{\Gamma}) \propto |\boldsymbol{\Gamma}|^{-\frac{\rho+m+1}{2}}\text{exp}\{-[\text{tr}(\mathbf{R})\boldsymbol{\Gamma}^{-1}/2]\} \qquad (3.13)$$

By substituting equations (3.3), (3.12) and (3.13) to (3.4)

$$\begin{aligned}\Pr(\boldsymbol{\theta},\boldsymbol{\Gamma} \mid \mathbf{S},\bar{\mathbf{r}},\mathbf{u}) &\propto |\boldsymbol{\Sigma}|^{-\frac{n}{2}}\exp[-\frac{n}{2}\mathrm{tr}(\mathbf{S}\boldsymbol{\Sigma}^{-1}) - \frac{n}{2}(\bar{\mathbf{r}}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{r}}-\boldsymbol{\mu})] \\ &\quad \times|\boldsymbol{\Gamma}|^{-\frac{\rho+m+2}{2}}\exp[\frac{-\mathrm{tr}(\mathbf{A}+\mathbf{R})\boldsymbol{\Gamma}^{-1}}{2}]\end{aligned}$$

and using the properties of the inverted Wishart probability density function obtained from Lee (1992),

$$\int |\boldsymbol{\Gamma}|^{-\frac{\rho+m+2}{2}}\exp[\frac{-\mathrm{tr}(\mathbf{A}+\mathbf{R})\boldsymbol{\Gamma}^{-1}}{2}]\ \mathrm{d}\boldsymbol{\Gamma} \propto |\mathbf{A}+\mathbf{R}|^{-(\frac{\rho+1}{2})}$$

As a result, the posterior density of $\boldsymbol{\theta}$ and the log-likelihood function can be expressed by the following equations:

$$\begin{aligned}\Pr(\boldsymbol{\theta} \mid \mathbf{S},\bar{\mathbf{r}},\mathbf{u}) &\propto |\boldsymbol{\Sigma}|^{-\frac{n}{2}}\exp[-\frac{n}{2}\mathrm{tr}(\mathbf{S}\boldsymbol{\Sigma}^{-1}) - \frac{n}{2}(\bar{\mathbf{r}}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{r}}-\boldsymbol{\mu})] \\ &\quad \times|\mathbf{A}+\mathbf{R}|^{-(\frac{\rho+1}{2})}\end{aligned}$$

and

$$\begin{aligned}\log\Pr(\boldsymbol{\theta} \mid \mathbf{S},\bar{\mathbf{r}},\mathbf{u}) &\propto -(\frac{n}{2})\log|\boldsymbol{\Sigma}| - \frac{n}{2}\mathrm{tr}(\mathbf{S}\boldsymbol{\Sigma}^{-1}) - \frac{n}{2}(\bar{\mathbf{r}}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{r}}-\boldsymbol{\mu}) \\ &\quad -(\frac{\rho+1}{2})\log|\mathbf{A}+\mathbf{R}| \\ &\propto -n[\log|\boldsymbol{\Sigma}| - \log|\mathbf{S}| + \mathrm{tr}(\mathbf{S}\boldsymbol{\Sigma}^{-1}) - p] - \mathbf{n}(\bar{\mathbf{r}}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{r}}-\boldsymbol{\mu}) \\ &\quad -(\rho+1)\log|\mathbf{A}+\mathbf{R}|\end{aligned}$$

The Bayesian estimate $\hat{\boldsymbol{\theta}}_3$ of $\boldsymbol{\theta}$ can be obtained by minimising the function:

$$F_3(\boldsymbol{\theta}) = F(\boldsymbol{\theta}) + B_3(\boldsymbol{\theta})$$

where F($\boldsymbol{\theta}$) is the same as that in equation (3.9) and is given by

$$F(\boldsymbol{\theta}) = n[\log|\boldsymbol{\Sigma}| - \log|\mathbf{S}| + \mathrm{tr}(\mathbf{S}\boldsymbol{\Sigma}^{-1}) - p] + n[(\bar{\mathbf{r}}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{r}}-\boldsymbol{\mu})]$$

and

$$B_3(\boldsymbol{\theta}) = (\rho + 1)\log|\mathbf{A} + \mathbf{R}|$$

3.4 Estimation of parameters using the Mx program

The estimation of $\hat{\boldsymbol{\theta}}_1$, $\hat{\boldsymbol{\theta}}_2$ and $\hat{\boldsymbol{\theta}}_3$ can be implemented using the Mx program (Neale et al., 1999). Mx is a structural equation modelling package. Similar to other major SEM packages, there are many built-in functions for structural equation modelling in Mx. Apart from that, it allows users to define their own fit functions. In our proposed model, the functions that are being minimised consist of two parts, $F(\boldsymbol{\theta})$ and $B_i(\boldsymbol{\theta})$. When n is sufficient large, $F(\boldsymbol{\theta})$ is nearly the same as the following built-in Maximum Likelihood fit function with mean structure of Mx:

$$(n-1)[\log|\boldsymbol{\Sigma}| - \log|\mathbf{S}| + \mathrm{tr}(\mathbf{S}\boldsymbol{\Sigma}^{-1}) - p] + n[(\bar{\mathbf{r}} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\bar{\mathbf{r}} - \boldsymbol{\mu})]$$

Therefore, this built-in fit function is used to specify $F(\boldsymbol{\theta})$. Moreover, three user defined fit functions are used to specify the $B_1(\boldsymbol{\theta})$, $B_2(\boldsymbol{\theta})$ and $B_3(\boldsymbol{\theta})$ respectively. Examples of Mx input script for obtaining $\hat{\boldsymbol{\theta}}_1$, $\hat{\boldsymbol{\theta}}_2$ and $\hat{\boldsymbol{\theta}}_3$ are given in Appendices D.1, D.2 and D.3 for reference. Note that each model consists of two groups. The first group is used to specify $F(\boldsymbol{\theta})$ and it is the same for all three cases. However, the second group is used to specify $B_i(\boldsymbol{\theta})$, so that they are set differently according to the form of $B_i(\boldsymbol{\theta})$.

Chapter 4

Empirical study on Hong Kong stock market

An empirical study based on a set of Hong Kong stock market data is carried out so as to illustrate how a single stage APT analysis with stochastic constraints on model parameters can be applied in practical situation.

4.1 Information of data

For the data of the study, we mainly consider constituent stocks of the Hong Kong Hang Seng Index (HSI). We have chosen 25 stocks from the constituent stock of 33 stocks, excluding stocks that are too new to provide enough data for conducting analysis and stocks with prices that have varied substantially due to some specific factors. The constituent stocks in the HSI are grouped into four main sectors: Utilities, Finance, Properties, and Commerce & Industry. Table 4.1 shows the chosen stocks and the respective sectors that they belong to.

4.2 Source of data

We got the data of the stocks from the data stream in the library of The Chinese University of Hong Kong. The time period of our data is from the 30^{th} December

Table 4.1: Stocks analysed in the study

Sector	Stock Code	Stocks
Finance (HSIF)	5	HSBC Holdings plc
	11	Hang Seng Bank Ltd.
	23	Bank of East Asia, Ltd.
Utilities (HSIU)	2	CLP Holdings Ltd.
	3	Hong Kong and China Gas Co. Ltd.
	6	Hong Kong Electric Holdings Ltd.
Properties (HSIP)	1	Cheung Kong (Holdings) Ltd.
	12	Henderson Land Development Co. Ltd.
	16	Sun Hung Kai Properties Ltd.
	20	Wheelock and Co. Ltd.
	97	Henderson Investment Ltd.
	101	Hang Lung Properties Ltd.
Commerce and Industry (HSIC&I)	4	Wharf (Holdings) Ltd.
	13	Hutchison Whampoa Ltd.
	19	Swire Pacific Ltd. 'A'
	179	Johnson Electric Holdings Ltd.
	267	CITIC Pacific Ltd.
	291	China Resources Enterprise, Ltd.
	293	Cathay Pacific Airways Ltd.
	330	Esprit Holdings Ltd.
	494	Li & Fung Ltd.
	511	Television Broadcasts Ltd.
	551	Yue Yuen Industrial (Holdings) Ltd.
	992	Legend Group Ltd.
	1199	COSCO Pacific Ltd.

1994 to 26^{th} December 2003. We downloaded the weekly closing price and the dividend of the stocks first, and then calculated the percentage of weekly return of each stock by the following equation:

$$r_t = \frac{P_t - P_{t-1} + D_t}{P_{t-1}} \times 100\%$$

where P_t stands for the stock price at time t and D_t is the dividend payment per share at the period between time t-1 and t. In total, we obtained 469 returns of each stock.

4.3 Lisrel model with exact constraints

We considered the use of a confirmatory factor analysis (CFA) with five latent factors, namely Hang Seng Index (HSI), Finance (HSIF), Utilities (HSIU), Properties (HSIP) and Commerce & Industry (HSIC&I) to analyse the APT model. HSI is a general factor that affects all the underlying stocks while the others are specific factors each corresponds to a sector of stocks. In the analysis with exact constraints, several parameters are fixed at the pre-assigned values.

Firstly, a reference variable for each latent factor is selected and their factor loadings are being fixed as the reference value of their corresponding factors. We call them "reference factor loadings" throughout the thesis. The selected reference variables and their value are tabulated in Table 4.2. They are obtained from an exploratory factor analysis, which was done prior the confirmatory factor analysis. From the exploratory factor analysis, for each latent factor, the observed variable with the largest factor loading is chosen to be the reference variable and its corresponding factor loading value is the reference value.

Table 4.2: Reference factor loadings

Factor	**Stock Code of Reference Variable**	**Reference Factor Loading**	**Reference value**
HSI	992	$b_{24,1}$	3.4800
HSIF	5	$b_{1,2}$	1.0900
HSIU	2	$b_{4,3}$	0.8810
HSIP	16	$b_{9,4}$	1.2300
HSIC&I	4	$b_{13,5}$	1.3900

Moreover, the variance of the general factor HSI and its covariance with other specific factors are being fixed to identify the latent factors. They are fixed at the value of the variance of the real HSI and the covariance of the real HSI with the four sub-indexes (HSIF, HSIU, HSIP and HSIC&I) obtained from the data stream in compliance with the period under study. Table 4.3 shows the covariance matrix of the real five indexes.

Table 4.3: Covariance matrix of five indexes

	HSI	HSIF	HSIU	HSIP	HSIC&I
HSI	**14.3605**				
HSIF	**12.3607**	14.1229			
HSIU	**6.3631**	5.6445	8.8835		
HSIP	**17.4215**	14.5177	7.0270	25.6698	
HSIC&I	**16.5198**	12.2312	6.2834	19.4052	21.1902

Remarks: The values in boldface are fixed

Lastly, we used a weekly deposit rate in bank as an approximate of the risk free rate. From the average of the percentage of weekly deposit rate in Hong Kong acquired from the data stream, it gives the risk free rate at a value of 0.0495%.

4.3.1 The resultant model

The estimated parameters of the factor loadings, factors covariance matrix, risk premia and the error covariance given by Mx are shown in Table 4.5.

From the estimation result, it is found that some stocks especially those belongs to commence & industry sector, load not only on the factor of their corresponding sector but also significantly on other factors. This result is reasonable since most of them are multi-industrial corporations.

For simplicity, the error covariance of the stocks is assumed zero. However, some stocks have close relationship among them, which means it is unreasonable to assume their error terms to be uncorrelated from each other. Therefore, the error covariance terms for some stocks are free to be estimated. Details are summarized in Table 4.4.

For the goodness of fit of the model, instead of reporting the chi-square statistic which is affected by sample size, several goodness of fit statistics given by Mx are considered. They are the Normed fit Index (NFI), Tucker Lewis Index (TLI), Relative Non-centrality Index (RNI) and Root Mean Squared Error Approximation (RMSEA). For the first three indexes, they are in range of zero to one, where zero indicates the model does not fit at all and one indicates perfect fit of the model. All of them should be greater than 0.95 for a good fit. For RMSEA, value below 0.05 indicates a very good fit. From Mx output, the NFI, TLI, RNI and RMSEA of our model are 0.98, 0.99, 0.99 and 0.041 respectively. These indicate a good fit of our model.

Table 4.4: Error covariance that are free to be estimated

Stocks	Stocks Code	Notation
HSBC , Hang Seng Bank	(5, 11)	$\Theta_\epsilon(2,1)$
CLP , HK Electric	(2, 6)	$\Theta_\epsilon(6,4)$
Cheung Kong, HK Electric	(1, 6)	$\Theta_\epsilon(7,6)$
HK&China Gas, Henderson Land	(3, 12)	$\Theta_\epsilon(8,5)$
HK&China Gas, Henderson Investment	(3, 97)	$\Theta_\epsilon(11,5)$
Henderson Land, Henderson Investment	(12, 97)	$\Theta_\epsilon(11,8)$
Wharf (Holdings), Wheelock	(4, 20)	$\Theta_\epsilon(13,10)$
Cheung Kong, Hutchison Whampoa	(1, 13)	$\Theta_\epsilon(14,7)$
CITIC Pacific, China Resources Enterprise	(267, 291)	$\Theta_\epsilon(18,17)$
Swire Pacific A, Cathay Pacific Airways	(19, 293)	$\Theta_\epsilon(19,15)$
Cathay Pacific Airways, CITIC Pacific	(293, 267)	$\Theta_\epsilon(19,17)$
China Resources Enterprise, COSCO Pacific	(291, 1199)	$\Theta_\epsilon(25,18)$

4.4 Lisrel model with stochastic constraints

In this section, we incorporate the stochastic constraints to the APT analysis. Instead of fixing the risk free rate and the factor loading references at their pre-assigned values, these fixed parameters will be free and the pre-assigned values will be regard as their prior information. This means we have the following stochastic constraints in our models.

$$
\begin{aligned}
0.0495 &= r_f + \epsilon_{u1} \\
3.4800 &= b_{24,1} + \epsilon_{u2} \\
1.0900 &= b_{1,2} + \epsilon_{u3} \\
0.8810 &= b_{4,3} + \epsilon_{u4} \\
1.2300 &= b_{9,4} + \epsilon_{u5} \\
1.3900 &= b_{13,5} + \epsilon_{u6}
\end{aligned}
$$

4.4.1 Result

For each model with different cases of stochastic constraints, several values are assigned to the prior constants to see how the Bayes estimates $\hat{\boldsymbol{\theta}}_1$, $\hat{\boldsymbol{\theta}}_2$ and $\hat{\boldsymbol{\theta}}_3$ are affected by the value of the parameters in the prior densities. Among all the parameters of the model, we have emphasized on the estimates of the risk free rate, reference factor loadings, risk premia and the latent factor covariance. Summary of estimation with stochastic constraints using Cases 1, 2 and 3 are tabulated in Tables 4.6, 4.9, and 4.12 respectively. For ease of comparison, the MLE estimates with exact constraints are given in the first column of each table. For reference, estimates of all factor loadings and error variances & covariance terms of the models are tabulated in Tables 4.7, 4.8, 4.10, 4.11, 4.13 and 4.14.

Table 4.6 presents the Bayes estimates $\hat{\boldsymbol{\theta}}_1$ for six sets of ν and β values. For the models 1A and 1B, the estimates especially for the risk free rate and the risk premia are significantly different from those achieved from the exact constraint model. When the value of ν becomes larger and β becomes smaller, the Bayes estimates get closer to those of the MLE estimates. In general, if the value of ν is sufficiently large and β is sufficiently small, then the Bayes estimates will be the same as the MLE estimates. Moreover, the result illustrates that the estimates of risk free rate and the risk premia are much sensitive to the changes of the prior constants' values than those of the factor loading references and the latent factor covariance.

Table 4.9 presents the Bayes estimates $\hat{\boldsymbol{\theta}}_2$ achieved from four sets of ν_i and β_i values. Here, we assume the variance of the error for the stochastic constraint on r_f (σ_1^2) is different from those on reference factor loadings (σ_2^2) that are all the same. Hence, we consider two sets of prior constants (ν_1, β_1) and (ν_2, β_2) for specifying the prior distribution of σ_1^2 and σ_2^2 respectively. In all four models, β_1s are different from β_2s while ν_1s equal ν_2s. It is found that there are similar observations as those of Case 1.

Similarly, Table 4.12 presents the Bayes estimates $\hat{\boldsymbol{\theta}}_3$ that are obtained with respect to eight sets combinations of the $\mathbf{R}$ matrices and ρ values. The following $\mathbf{R}$ matrices are considered.

$$\mathbf{R1} = \begin{pmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.5 & 0.5 & 0.5 & 0.5 \\ 0.0 & 0.5 & 1.0 & 0.5 & 0.5 & 0.5 \\ 0.0 & 0.5 & 0.5 & 1.0 & 0.5 & 0.5 \\ 0.0 & 0.5 & 0.5 & 0.5 & 1.0 & 0.5 \\ 0.0 & 0.5 & 0.5 & 0.5 & 0.5 & 1.0 \end{pmatrix} \mathbf{R2} = \begin{pmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \end{pmatrix}$$

In both matrices, the error for the constraint on r_f is assumed to be uncorrelated with those on the factor loading references. In matrix $\mathbf{R1}$, the errors for the constraints corresponding to the factor loading references are considered to be dependent. In matrix $\mathbf{R2}$, they are considered to be independent. For the models 3A to 3D (see Table 4.12), we found that when the ρ value is small, the Bayes estimates have large differences from those of MLE estimates. In addition, when comparing model 3A with 3B and model 3C with 3D, discrepancies in the estimates are found. Particularly, the estimates of model 3A with $\mathbf{R}$=$\mathbf{R1}$ have larger differences from the MLE estimates than that of model 3B with $\mathbf{R}$=$\mathbf{R2}$. When the value of ρ gets larger, the Bayes estimates are nearly the same as the MLE ones and the discrepancies between the models using different $\mathbf{R}$ matrices disappear.

From the result of models 1A, 1B, 2A, 2B, 3A and 3B in Table 4.6, 4.9, and 4.12, we can see that the estimate of r_f is extraordinary large and most of the estimates of the Market Risk Premium are negative, these estimation results are not compatible with the reality. In practical situation, it is worthy to note that when the prior constants are chosen be certain values such that it leads to the non-information prior distributions, which means there is little prior information about the parameters with stochastic constraints. While traditional APT analysis operates on the assumption that the risk-free rate is known, the incorporation of stochastic constraint so to allow variation in the risk-free rate to an extent that the estimate provided is too large and not compatible with the reality will lead to unreasonable estimate of the factor risk premia as well.

Table 4.5: Empirical result - Model E (Exact constraints)

Factor loadings

Stock/Index	HSI	HSIF	HSIU	HSIP	HSIC&I
5	-0.1388	**1.0900**	0	0	0
11	-2.5733	3.7520	0	0	0
23	-2.6925	4.1053	0	0	0
2	-0.1436	0	**0.8810**	0	0
3	-0.0649	0	0.9368	0	0
6	-0.1604	0	0.9068	0	0
1	0.7274	0	0	0.4968	0
12	-0.6877	0	0	1.4797	0
16	-0.1614	0	0	**1.2300**	0
20	-2.9918	4.2658	0	0.2988	0
97	0.1059	0	0	0.6015	0
101	-0.4023	0	0	0.9443	0
4	-3.4574	3.3829	0	0	**1.3900**
13	0.7728	0	0	0	0.3644
19	-2.2208	0	0	1.0796	1.6028
179	-0.6933	-5.8844	0	0.0517	5.5055
267	-0.6590	0	0	0	1.4785
291	-2.1167	0	0	0.3217	2.5876
293	-2.7036	0	0	0	2.8763
330	-3.7587	-5.0725	0	0.3486	7.2672
494	1.0717	-7.6323	0	0	5.3056
511	1.0038	-2.3766	0	0	1.5715
551	-1.4631	0	0	0	1.5925
992	**3.4800**	-9.1480	0	0	4.8563
1199	-2.9204	0	0	0	3.4151

Market risk premium

HSNI	HSNF	HSNU	HSNP	HSNC&I
0.3195	0.2551	0.2788	0.2644	0.4243

r_f **0.0495**

Latent factors covariance matrix

	HSI	HSIF	HSIU	HSIP	HSNC&I
HSI	**14.3605**	*	*	*	*
HSIF	**12.3607**	10.8981	*	*	*
HSIU	**6.3631**	6.0070	6.6329	*	*
HSIP	**17.4215**	15.4995	8.8123	24.4095	*
HSNC&I	**16.5198**	14.4978	7.8517	20.4015	19.5195

Error covariance of the stocks

Stock	Variance	Stock	Variance	Stocks	Covariance
5	6.9486	13	7.5546	5, 11	1.4575
11	5.2507	19	12.2074	2, 6	1.4927
23	9.1033	179	25.7166	1, 6	0.3845
2	5.8093	267	14.7645	3, 12	0.0283
3	4.9639	291	28.0468	3, 97	2.3114
6	4.2824	293	17.2525	12, 97	-0.6303
1	5.5819	330	28.8383	4, 20	8.4350
12	6.6507	494	25.5322	1, 13	3.1800
16	5.2989	511	23.4554	19, 293	1.8727
20	14.0674	551	22.2282	293, 267	-0.6215
97	16.3770	992	61.5338	267, 291	4.4152
101	12.1591	1199	34.6262	291, 1199	8.5309
4	16.7540				

Remarks: The values in boldface are fixed

Table 4.6: Empirical result - Exact constraints vs Stochastic Case 1

Model	E	1A	1B	1C	1D	1E	1F
	Exact	ν=5	ν=50	ν=5	ν=5	ν=50	ν=500
		β=0.1	β=0.1	β=0.01	β=10^{-3}	β=10^{-4}	β=10^{-5}
Constraints							
r_f	0.050	0.380	0.359	0.104	0.055	0.051	0.050
$b_{24,1}$	3.480	3.351	3.365	3.474	3.480	3.480	3.480
$b_{1,2}$	1.090	1.564	1.558	1.131	1.094	1.091	1.090
$b_{4,3}$	0.881	0.880	0.881	0.881	0.881	0.881	0.881
$b_{9,4}$	1.230	1.230	1.230	1.230	1.230	1.230	1.230
$b_{13,5}$	1.390	0.544	0.761	1.366	1.388	1.390	1.390
Market risk premium (λ_i)							
HSI	0.320	0.031	0.048	0.271	0.315	0.319	0.319
HSIF	0.255	-0.020	-0.002	0.210	0.250	0.254	0.255
HSIU	0.279	-0.120	-0.093	0.215	0.272	0.278	0.279
HSIP	0.264	-0.079	-0.056	0.209	0.259	0.263	0.264
HSIC&I	0.424	0.070	0.087	0.364	0.418	0.423	0.424
Coavariance of latent factors							
HSIF	10.898	11.202	11.126	10.914	10.900	10.898	10.898
HSIU	6.633	7.082	7.029	6.671	6.636	6.633	6.633
HSIP	24.410	24.943	24.900	24.439	24.412	24.410	24.410
HSIC&I	19.520	21.349	20.700	19.574	19.524	19.520	19.520
HSIF, HSIU	6.007	6.363	6.291	6.030	6.009	6.007	6.007
HSIF, HSIP	15.500	15.983	15.892	15.529	15.502	15.500	15.500
HSIF, HSIC&I	14.498	15.154	14.962	14.525	14.500	14.498	14.498
HSIU, HSIP	8.812	9.421	9.349	8.858	8.817	8.813	8.812
HSIU, HSIC&I	7.852	8.693	8.479	7.892	7.855	7.852	7.852
HSIP, HSIC&I	20.402	21.393	21.171	20.449	20.406	20.402	20.402

Table 4.7: Empirical result - Exact constraints vs Stochastic Case 1

Model	E	1A	1B	1C	1D	1E	1F
Factor loadings							
$b_{1,1}$	-0.1388	-0.5925	-0.5815	-0.1771	-0.1423	-0.1395	-0.1389
$b_{2,1}$	-2.5733	-1.8874	-2.0295	-2.5124	-2.5674	-2.5721	-2.5732
$b_{3,1}$	-2.6925	-2.0919	-2.2437	-2.6454	-2.6880	-2.6917	-2.6925
$b_{4,1}$	-0.1436	-0.1757	-0.1723	-0.1461	-0.1439	-0.1437	-0.1436
$b_{5,1}$	-0.0649	-0.0831	-0.0805	-0.0659	-0.0651	-0.0650	-0.0650
$b_{6,1}$	-0.1604	-0.1790	-0.1768	-0.1617	-0.1605	-0.1604	-0.1604
$b_{7,1}$	0.7274	0.7529	0.7509	0.7319	0.7279	0.7275	0.7274
$b_{8,1}$	-0.6877	-0.6856	-0.6890	-0.6916	-0.6881	-0.6878	-0.6877
$b_{9,1}$	-0.1614	-0.1856	-0.1833	-0.1626	-0.1615	-0.1614	-0.1614
$b_{10,1}$	-2.9918	-2.2034	-2.3715	-2.9279	-2.9856	-2.9905	-2.9917
$b_{11,1}$	0.1059	-0.1762	-0.1426	0.0793	0.1033	0.1053	0.1058
$b_{12,1}$	-0.4023	-0.5938	-0.5771	-0.4238	-0.4045	-0.4028	-0.4024
$b_{13,1}$	-3.4574	-2.1145	-2.3356	-3.3400	-3.4461	-3.4551	-3.4571
$b_{14,1}$	0.7728	0.7991	0.7584	0.7651	0.7720	0.7727	0.7728
$b_{15,1}$	-2.2208	-1.2560	-1.3965	-2.1288	-2.2119	-2.2190	-2.2207
$b_{16,1}$	-0.6933	0.8264	0.5927	-0.5565	-0.6799	-0.6906	-0.6928
$b_{17,1}$	-0.6590	-0.0908	-0.2309	-0.6193	-0.6553	-0.6583	-0.6589
$b_{18,1}$	-2.1167	-0.9854	-1.2196	-2.0311	-2.1086	-2.1153	-2.1165
$b_{19,1}$	-2.7036	-1.1855	-1.4644	-2.5783	-2.6915	-2.7012	-2.7034
$b_{20,1}$	-3.7587	-0.7652	-1.2051	-3.4796	-3.7318	-3.7534	-3.7580
$b_{21,1}$	1.0717	2.0281	1.8796	1.1499	1.0792	1.0732	1.0719
$b_{22,1}$	1.0038	0.8907	0.8508	0.9720	1.0007	1.0032	1.0038
$b_{23,1}$	-1.4631	-0.5617	-0.7028	-1.3696	-1.4539	-1.4613	-1.4629
$b_{24,1}$	**3.4800**	3.3509	3.3652	3.4741	3.4795	3.4799	3.4800
$b_{25,1}$	-2.9204	-1.2232	-1.5425	-2.7779	-2.9067	-2.9179	-2.9201
$b_{1,2}$	**1.0900**	1.5637	1.5581	1.1305	1.0937	1.0907	1.0901
$b_{2,2}$	3.7520	2.8931	3.0651	3.6761	3.7447	3.7506	3.7519
$b_{3,2}$	4.1053	3.3229	3.5092	4.0444	4.0995	4.1042	4.1052
$b_{10,2}$	4.2658	3.1580	3.3916	4.1821	4.2578	4.2642	4.2657
$b_{13,2}$	3.3829	2.8818	2.8568	3.2733	3.3722	3.3807	3.3827
$b_{16,2}$	-5.8844	-4.1684	-4.6159	-5.7888	-5.8759	-5.8832	-5.8849
$b_{20,2}$	-5.0725	-3.9531	-4.2861	-5.051	-5.0707	-5.0728	-5.0729
$b_{21,2}$	-7.6323	-5.5439	-6.1030	-7.5089	-7.6201	-7.6299	-7.6319
$b_{22,2}$	-2.3766	-1.5765	-1.7550	-2.3110	-2.3702	-2.3752	-2.3764
$b_{24,2}$	-9.1480	-6.6728	-7.3827	-9.0244	-9.1364	-9.1457	-9.1477
$b_{4,3}$	**0.8810**	0.8802	0.8813	0.8810	0.8810	0.8810	0.8810
$b_{5,3}$	0.9368	0.9127	0.9145	0.9339	0.9365	0.9368	0.9368
$b_{6,3}$	0.9068	0.8868	0.8895	0.9050	0.9067	0.9068	0.9068
$b_{7,4}$	0.4968	0.4954	0.4952	0.4951	0.4966	0.4967	0.4968
$b_{8,4}$	1.4797	1.4490	1.4541	1.4807	1.4799	1.4798	1.4797
$b_{9,4}$	**1.2300**	1.2298	1.2300	1.2300	1.2300	1.2300	1.2300
$b_{10,4}$	0.2988	0.3765	0.3563	0.3016	0.2991	0.2989	0.2988
$b_{11,4}$	0.6015	0.7925	0.7700	0.6201	0.6033	0.6019	0.6016
$b_{12,4}$	0.9443	1.0579	1.0482	0.9581	0.9457	0.9446	0.9443
$b_{15,4}$	1.0796	1.0946	1.0814	1.0774	1.0794	1.0796	1.0796
$b_{16,4}$	0.0517	0.0294	0.0202	0.0459	0.0512	0.0516	0.0518
$b_{18,4}$	0.3217	0.4007	0.3816	0.3254	0.3220	0.3217	0.3217
$b_{20,4}$	0.3486	0.3681	0.3353	0.3444	0.3483	0.3486	0.3487
$b_{13,5}$	**1.3900**	0.5444	0.7608	1.3663	1.3879	1.3896	1.3900
$b_{14,5}$	0.3644	0.3529	0.3883	0.3728	0.3653	0.3646	0.3644
$b_{15,5}$	1.6028	0.7008	0.8418	1.5219	1.5950	1.6012	1.6027
$b_{16,5}$	5.5055	2.8926	3.4442	5.3193	5.4879	5.5024	5.5055
$b_{17,5}$	1.4785	0.9597	1.0853	1.4430	1.4751	1.4778	1.4784
$b_{18,5}$	2.5876	1.4496	1.6819	2.5044	2.5797	2.5862	2.5875
$b_{19,5}$	2.8763	1.5069	1.7544	2.7635	2.8654	2.8741	2.8761
$b_{20,5}$	7.2672	3.7356	4.4103	7.0079	7.2423	7.2627	7.2668
$b_{21,5}$	5.3056	2.8983	3.4467	5.1441	5.2898	5.3026	5.3051
$b_{22,5}$	1.5715	1.0514	1.2227	1.5492	1.5694	1.5709	1.5714
$b_{23,5}$	1.5925	0.7817	0.9068	1.5082	1.5842	1.5909	1.5923
$b_{24,5}$	4.8563	3.1020	3.6274	4.7703	4.8482	4.8547	4.8561
$b_{25,5}$	3.4151	1.8734	2.1590	3.2867	3.4028	3.4128	3.4149

Table 4.8: Empirical result - Exact constraints vs Stochastic Case 1

Parameters	E	1A	1B	1C	1D	1E	1F
Error variance and covariance							
$\Theta_\epsilon(1,1)$	6.9486	6.8501	6.8630	6.9425	6.9481	6.9486	6.9487
$\Theta_\epsilon(2,2)$	5.2507	5.2306	5.2381	5.2503	5.2507	5.2508	5.2508
$\Theta_\epsilon(3,3)$	9.1033	8.9266	8.9529	9.0931	9.1024	9.1031	9.1033
$\Theta_\epsilon(4,4)$	5.8093	5.7051	5.7127	5.7993	5.8085	5.8093	5.8093
$\Theta_\epsilon(5,5)$	4.9639	5.0235	5.0210	4.9711	4.9644	4.9638	4.9638
$\Theta_\epsilon(6,6)$	4.2824	4.2474	4.2495	4.2787	4.2822	4.2825	4.2824
$\Theta_\epsilon(7,7)$	5.5819	4.5529	4.6661	5.5008	5.5742	5.5804	5.5818
$\Theta_\epsilon(8,8)$	6.6507	6.9273	6.8790	6.6626	6.6517	6.6509	6.6507
$\Theta_\epsilon(9,9)$	5.2989	5.4175	5.3956	5.3076	5.2998	5.2991	5.2989
$\Theta_\epsilon(10,10)$	14.0674	13.9523	13.9705	14.0558	14.0663	14.067	14.0672
$\Theta_\epsilon(11,11)$	16.3770	16.2754	16.2975	16.3718	16.3765	16.3768	16.3769
$\Theta_\epsilon(12,12)$	12.1591	11.8657	11.8936	12.1308	12.1564	12.1586	12.1592
$\Theta_\epsilon(13,13)$	16.7540	16.7338	16.7762	16.7597	16.7546	16.7541	16.7539
$\Theta_\epsilon(14,14)$	7.5546	6.9209	6.9686	7.4987	7.5493	7.5535	7.5545
$\Theta_\epsilon(15,15)$	12.2074	12.2782	12.2689	12.2147	12.2082	12.2078	12.2072
$\Theta_\epsilon(16,16)$	25.7166	25.9544	25.8695	25.7275	25.7167	25.7159	25.7158
$\Theta_\epsilon(17,17)$	14.7645	14.7024	14.7365	14.7638	14.7644	14.7641	14.7646
$\Theta_\epsilon(18,18)$	28.0468	28.1251	28.1211	28.0571	28.0477	28.0470	28.0471
$\Theta_\epsilon(19,19)$	17.2525	17.2401	17.2491	17.2523	17.2525	17.2529	17.2524
$\Theta_\epsilon(20,20)$	28.8383	28.8710	28.9526	28.8351	28.8390	28.8387	28.8391
$\Theta_\epsilon(21,21)$	25.5322	25.6912	25.6139	25.5385	25.5336	25.5337	25.5332
$\Theta_\epsilon(22,22)$	23.4554	23.7959	23.7610	23.4897	23.4588	23.4566	23.4557
$\Theta_\epsilon(23,23)$	22.2282	22.1853	22.2079	22.2266	22.2277	22.2287	22.2283
$\Theta_\epsilon(24,24)$	61.5338	61.9335	61.7541	61.5083	61.5310	61.5332	61.5336
$\Theta_\epsilon(25,25)$	34.6262	34.5231	34.6167	34.6415	34.6279	34.6269	34.6266
$\Theta_\epsilon(2,1)$	1.4575	1.1902	1.2274	1.4402	1.4560	1.4573	1.4575
$\Theta_\epsilon(6,4)$	1.4927	1.4177	1.4233	1.4855	1.4922	1.4928	1.4927
$\Theta_\epsilon(7,6)$	0.3845	0.4098	0.4048	0.3861	0.3847	0.3846	0.3845
$\Theta_\epsilon(8,5)$	0.0283	-0.0137	-0.0068	0.0241	0.0278	0.0281	0.0282
$\Theta_\epsilon(11,5)$	2.3114	2.2963	2.2986	2.3107	2.3113	2.3113	2.3114
$\Theta_\epsilon(11,8)$	-0.6303	-0.9181	-0.8864	-0.6610	-0.6334	-0.6311	-0.6305
$\Theta_\epsilon(13,10)$	8.4350	8.3528	8.3932	8.4348	8.4350	8.4349	8.4349
$\Theta_\epsilon(14,7)$	3.1800	2.3465	2.4253	3.1093	3.1733	3.1786	3.1798
$\Theta_\epsilon(18,17)$	1.8727	1.8804	1.8611	1.8657	1.8720	1.8728	1.8725
$\Theta_\epsilon(19,15)$	-0.6215	-0.6168	-0.5797	-0.6079	-0.6201	-0.6212	-0.6214
$\Theta_\epsilon(19,17)$	4.4152	4.5422	4.5376	4.4265	4.4162	4.4150	4.4153
$\Theta_\epsilon(25,18)$	8.5309	8.5022	8.5257	8.5353	8.5314	8.5313	8.5310

Table 4.9: Empirical result - Exact constraints vs Stochastic Case 2

Model	E Exact	2A $\nu_1, \nu_2 = 5$ $\beta_1 = 1$ $\beta_2 = 0.1$	2B $\nu_1, \nu_2 = 5$ $\beta_1 = 0.1$ $\beta_2 = 0.01$	2C $\nu_1, \nu_2 = 50$ $\beta_1 = 0.01$ $\beta_2 = 10^{-3}$	2D $\nu_1, \nu_2 = 500$ $\beta_1 = 10^{-3}$ $\beta_2 = 10^{-4}$
Constraints					
r_f	0.050	0.443	0.350	0.144	0.062
ref HSI	3.480	3.350	3.474	3.479	3.480
ref HSIF	1.090	1.557	1.134	1.097	1.091
ref HSIU	0.881	0.880	0.881	0.881	0.881
ref HSIP	1.230	1.230	1.230	1.230	1.230
ref HSIC&I	1.390	0.496	1.364	1.386	1.390
Market risk premium (λ_i)					
HSI	0.320	-0.016	0.063	0.238	0.309
HSIF	0.255	-0.069	0.013	0.179	0.245
HSIU	0.279	-0.198	-0.083	0.167	0.265
HSIP	0.264	-0.143	-0.047	0.167	0.252
HSIC&I	0.424	-0.004	0.089	0.319	0.411
Covariance of latent factors					
HSIF	10.898	11.223	10.927	10.905	10.899
HSIU	6.633	7.136	6.819	6.672	6.637
HSIP	24.410	24.939	24.426	24.407	24.408
HSIC&I	19.520	21.558	19.610	19.541	19.522
HSIF, HSIU	6.007	6.386	6.056	6.019	6.008
HSIF, HSIP	15.500	16.005	15.548	15.512	15.501
HSIF, HSIC&I	14.498	15.210	14.542	14.508	14.499
HSIU, HSIP	8.812	9.441	8.904	8.837	8.815
HSIU, HSIC&I	7.852	8.763	7.930	7.871	7.854
HSIP, HSIC&I	20.402	21.449	20.473	20.419	20.403

Table 4.10: Empirical result - Exact constraints vs Stochastic Case 2

Parameters	E	2A	2B	2C	2D
Factor loadings					
$b_{1,1}$	-0.1388	-0.5887	-0.1855	-0.1481	-0.1399
$b_{2,1}$	-2.5733	-1.8498	-2.4648	-2.5463	-2.5706
$b_{3,1}$	-2.6925	-2.0483	-2.6026	-2.6739	-2.6909
$b_{4,1}$	-0.1436	-0.1765	-0.1482	-0.1450	-0.1438
$b_{5,1}$	-0.0649	-0.0797	-0.0547	-0.0635	-0.0649
$b_{6,1}$	-0.1604	-0.1796	-0.1626	-0.1611	-0.1605
$b_{7,1}$	0.7274	0.7538	0.7403	0.7321	0.7280
$b_{8,1}$	-0.6877	-0.6853	-0.6973	-0.6914	-0.6882
$b_{9,1}$	-0.1614	-0.1858	-0.1622	-0.1612	-0.1613
$b_{10,1}$	-2.9918	-2.1601	-2.8843	-2.9682	-2.9896
$b_{11,1}$	0.1059	-0.1850	0.0490	0.0867	0.1032
$b_{12,1}$	-0.4023	-0.5989	-0.4507	-0.4183	-0.4044
$b_{13,1}$	-3.4574	-2.0582	-3.2579	-3.4101	-3.4528
$b_{14,1}$	0.7728	0.8096	0.7504	0.7653	0.7718
$b_{15,1}$	-2.2208	-1.2261	-2.0912	-2.1860	-2.2173
$b_{16,1}$	-0.6933	0.8761	-0.4960	-0.6362	-0.6875
$b_{17,1}$	-0.6590	-0.0553	-0.6189	-0.6548	-0.6597
$b_{18,1}$	-2.1167	-0.9213	-1.9698	-2.0871	-2.1151
$b_{19,1}$	-2.7036	-1.1204	-2.5279	-2.6574	-2.6993
$b_{20,1}$	-3.7587	-0.6609	-3.3455	-3.6513	-3.7501
$b_{21,1}$	1.0717	2.0566	1.1813	1.1071	1.0754
$b_{22,1}$	1.0038	0.8944	0.9100	0.9782	1.0011
$b_{23,1}$	-1.4631	-0.5237	-1.2843	-1.4099	-1.4573
$b_{24,1}$	**3.4800**	3.3496	3.4735	3.4790	3.4799
$b_{25,1}$	-2.9204	-1.1397	-2.6936	-2.8696	-2.9171
$b_{1,2}$	**1.0900**	1.5570	1.1340	1.0974	1.0908
$b_{2,2}$	3.7520	2.8480	3.6159	3.7175	3.7486
$b_{3,2}$	4.1053	3.2706	3.9904	4.0807	4.1031
$b_{10,2}$	4.2658	3.0994	4.1297	4.2356	4.2630
$b_{13,2}$	3.3829	2.8806	3.1804	3.3329	3.3782
$b_{16,2}$	-5.8844	-4.0430	-5.6454	-5.8197	-5.8753
$b_{20,2}$	-5.0725	-3.8846	-5.0902	-5.1015	-5.0832
$b_{21,2}$	-7.6323	-5.3821	-7.2982	-7.5358	-7.6185
$b_{22,2}$	-2.3766	-1.5196	-2.1973	-2.3226	-2.3694
$b_{24,2}$	-9.1480	-6.4812	-8.8850	-9.0846	-9.1421
$b_{4,3}$	**0.8810**	0.8800	0.8810	0.8810	0.8810
$b_{5,3}$	0.9368	0.9031	0.9038	0.9306	0.9362
$b_{6,3}$	0.9068	0.8862	0.9029	0.9058	0.9067
$b_{7,4}$	0.4968	0.4950	0.4909	0.4945	0.4964
$b_{8,4}$	1.4797	1.4488	1.4849	1.4822	1.4801
$b_{9,4}$	**1.2300**	1.2298	1.2300	1.2300	1.2300
$b_{10,4}$	0.2988	0.3814	0.3016	0.2997	0.2989
$b_{11,4}$	0.6015	0.7986	0.6412	0.6149	0.6034
$b_{12,4}$	0.9443	1.0614	0.9763	0.9549	0.9457
$b_{15,4}$	1.0796	1.0987	1.0799	1.0807	1.0800
$b_{16,4}$	0.0517	0.0320	0.0441	0.0504	0.0517
$b_{18,4}$	0.3217	0.4054	0.3287	0.3240	0.3221
$b_{20,4}$	0.3486	0.3824	0.3618	0.3565	0.3511
$b_{13,5}$	**1.3900**	0.4958	1.3643	1.3858	1.3896
$b_{14,5}$	0.3644	0.3437	0.3875	0.3722	0.3655
$b_{15,5}$	1.6028	0.6699	1.4860	1.5707	1.5994
$b_{16,5}$	5.5055	2.7522	5.1607	5.4082	5.4937
$b_{17,5}$	1.4785	0.9285	1.4439	1.4750	1.4791
$b_{18,5}$	2.5876	1.3871	2.4439	2.5574	2.5856
$b_{19,5}$	2.8763	1.4498	2.7187	2.8349	2.8724
$b_{20,5}$	7.2672	3.5770	6.8989	7.1851	7.2648
$b_{21,5}$	5.3056	2.7522	4.9569	5.2015	5.2919
$b_{22,5}$	1.5715	1.0054	1.5183	1.5530	1.5684
$b_{23,5}$	1.5925	0.7478	1.4288	1.5433	1.5871
$b_{24,5}$	4.8563	2.9577	4.6659	4.8099	4.8520
$b_{25,5}$	3.4151	1.7991	3.2099	3.3689	3.4120

Table 4.11: Empirical result - Exact constraints vs Stochastic Case 2

Parameters	E	2A	2B	2C	2D
Error variance and covariance					
$\Theta_\epsilon(1,1)$	6.9486	6.8420	6.9247	6.9425	6.9480
$\Theta_\epsilon(2,2)$	5.2507	5.2272	5.2453	5.2495	5.2506
$\Theta_\epsilon(3,3)$	9.1033	8.9248	9.0973	9.1009	9.1034
$\Theta_\epsilon(4,4)$	5.8093	5.6776	5.7172	5.7925	5.8077
$\Theta_\epsilon(5,5)$	4.9639	5.0519	5.0601	4.9820	4.9656
$\Theta_\epsilon(6,6)$	4.2824	4.2219	4.2030	4.2690	4.2812
$\Theta_\epsilon(7,7)$	5.5819	4.5306	5.4266	5.5358	5.5767
$\Theta_\epsilon(8,8)$	6.6507	6.9387	6.6747	6.6571	6.6517
$\Theta_\epsilon(9,9)$	5.2989	5.4245	5.3221	5.3064	5.3000
$\Theta_\epsilon(10,10)$	14.0674	13.9434	14.0267	14.0539	14.0655
$\Theta_\epsilon(11,11)$	16.3770	16.2685	16.3605	16.3702	16.3761
$\Theta_\epsilon(12,12)$	12.1591	11.8588	12.0924	12.1359	12.1559
$\Theta_\epsilon(13,13)$	16.7540	16.7229	16.7552	16.7495	16.7531
$\Theta_\epsilon(14,14)$	7.5546	6.9197	7.4696	7.5308	7.5524
$\Theta_\epsilon(15,15)$	12.2074	12.2771	12.2124	12.2116	12.2082
$\Theta_\epsilon(16,16)$	25.7166	25.9844	25.7807	25.7437	25.7225
$\Theta_\epsilon(17,17)$	14.7645	14.6874	14.7465	14.7570	14.7631
$\Theta_\epsilon(18,18)$	28.0468	28.1481	28.1162	28.0587	28.0480
$\Theta_\epsilon(19,19)$	17.2525	17.2269	17.2288	17.2496	17.2524
$\Theta_\epsilon(20,20)$	28.8383	28.8177	28.6981	28.7717	28.8165
$\Theta_\epsilon(21,21)$	25.5322	25.7221	25.5993	25.5628	25.5419
$\Theta_\epsilon(22,22)$	23.4554	23.8153	23.5718	23.4897	23.4592
$\Theta_\epsilon(23,23)$	22.2282	22.1726	22.1798	22.2096	22.2264
$\Theta_\epsilon(24,24)$	61.5338	61.9805	61.4702	61.5151	61.5223
$\Theta_\epsilon(25,25)$	34.6262	34.5167	34.6717	34.6253	34.6219
$\Theta_\epsilon(2,1)$	1.4575	1.1786	1.4193	1.4487	1.4565
$\Theta_\epsilon(6,4)$	1.4927	1.3909	1.4061	1.4774	1.4913
$\Theta_\epsilon(7,6)$	0.3845	0.4131	0.3923	0.3865	0.3848
$\Theta_\epsilon(8,5)$	0.0283	-0.0144	0.0205	0.0247	0.0276
$\Theta_\epsilon(11,5)$	2.3114	2.2969	2.3141	2.3117	2.3115
$\Theta_\epsilon(11,8)$	-0.6303	-0.9261	-0.6974	-0.654	-0.6341
$\Theta_\epsilon(13,10)$	8.4350	8.3403	8.4209	8.4279	8.4338
$\Theta_\epsilon(14,7)$	3.1800	2.3349	3.0524	3.1422	3.1758
$\Theta_\epsilon(18,17)$	1.8727	1.878	1.8435	1.8676	1.8727
$\Theta_\epsilon(19,15)$	-0.6215	-0.6321	-0.6158	-0.6219	-0.6220
$\Theta_\epsilon(19,17)$	4.4152	4.5368	4.4159	4.4125	4.4145
$\Theta_\epsilon(25,18)$	8.5309	8.5197	8.5865	8.5380	8.5302

Table 4.12: Empirical result - Exact constraints vs Stochastic Case 3

Model	E Exact	3A ρ= 5 R = R1	3B ρ= 5 R = R2	3C ρ= 50 R = R1	3D ρ= 50 R = R2	3E ρ= 500 R = R1	3F ρ= 500 R = R2	3G ρ= 5000 R = R1	3H ρ= 5000 R = R2
Constraints									
r_f	0.050	0.427	0.426	0.206	0.209	0.073	0.073	0.052	0.052
$b_{24,1}$	3.480	2.755	3.407	3.486	3.453	3.481	3.478	3.480	3.480
$b_{1,2}$	1.090	0.620	1.306	1.191	1.254	1.101	1.106	1.091	1.092
$b_{4,3}$	0.881	0.155	0.881	0.900	0.881	0.883	0.881	0.881	0.881
$b_{9,4}$	1.230	0.568	1.230	1.249	1.230	1.232	1.230	1.230	1.230
$b_{13,5}$	1.390	0.008	0.091	1.358	1.278	1.388	1.381	1.390	1.389
Market risk premium (λ_i)									
HSI	0.320	-0.003	-0.003	0.183	0.177	0.299	0.299	0.317	0.317
HSIF	0.255	-0.143	-0.069	0.128	0.123	0.236	0.236	0.253	0.253
HSIU	0.279	-1.003	-0.176	0.092	0.089	0.251	0.251	0.276	0.276
HSIP	0.264	-0.268	-0.125	0.104	0.100	0.241	0.240	0.262	0.262
HSIC&I	0.424	0.350	0.052	0.249	0.246	0.398	0.398	0.422	0.422
Covariance of latent factors									
HSIF	10.898	15.043	11.637	10.927	10.964	10.901	10.905	10.898	10.899
HSIU	6.633	143.358	7.150	6.580	6.781	6.624	6.648	6.632	6.634
HSIP	24.410	38.110	24.837	24.374	24.553	24.402	24.421	24.409	24.411
HSIC&I	19.520	450.669	29.425	19.627	19.762	19.531	19.541	19.521	19.522
HSIF, HSIU	6.007	19.765	6.672	6.041	6.097	6.011	6.016	6.007	6.008
HSIF, HSIP	15.500	20.959	16.324	15.549	15.617	15.505	15.511	15.500	15.501
HSIF, HSIC&I	14.498	49.106	16.799	14.549	14.613	14.503	14.509	14.498	14.499
HSIU, HSIP	8.812	29.516	9.499	8.874	8.990	8.819	8.831	8.813	8.814
HSIU, HSIC&I	7.852	112.877	10.206	7.924	8.021	7.859	7.868	7.853	7.853
HSIP, HSIC&I	20.402	55.897	22.715	20.495	20.607	20.411	20.420	20.402	20.403

Table 4.13: Empirical result - Exact constraints vs Stochastic Case 3

Parameters	E	3A	3B	3C	3D	3E	3F	3G	3H
Factor loadings									
$b_{1,1}$	-0.1388	0.2100	-0.3806	-0.2335	-0.2924	-0.1489	-0.1541	-0.1398	-0.1404
$b_{2,1}$	-2.5733	-0.3085	-1.3089	-2.4833	-2.3589	-2.5631	-2.5482	-2.5722	-2.5707
$b_{3,1}$	-2.6925	-0.2850	-1.4384	-2.6352	-2.5257	-2.6868	-2.6733	-2.6919	-2.6906
$b_{4,1}$	-0.1436	0.1405	-0.1813	-0.1575	-0.1532	-0.1454	-0.1446	-0.1438	-0.1437
$b_{5,1}$	-0.0649	0.2456	-0.0870	-0.0734	-0.0685	-0.0662	-0.0654	-0.0651	-0.0650
$b_{6,1}$	-0.1604	0.1394	-0.1840	-0.1718	-0.1654	-0.1618	-0.1609	-0.1605	-0.1605
$b_{7,1}$	0.7274	1.0774	0.7540	0.7285	0.7408	0.7278	0.7294	0.7274	0.7276
$b_{8,1}$	-0.6877	0.2623	-0.6868	-0.7230	-0.6969	-0.6926	-0.6895	-0.6882	-0.6879
$b_{9,1}$	-0.1614	0.6228	-0.1840	-0.1867	-0.1672	-0.1645	-0.1618	-0.1617	-0.1614
$b_{10,1}$	-2.9918	-0.2356	-1.5719	-2.9074	-2.7608	-2.9830	-2.9656	-2.9908	-2.9891
$b_{11,1}$	0.1059	0.2958	-0.2505	0.0347	0.0128	0.0966	0.0949	0.1049	0.1047
$b_{12,1}$	-0.4023	0.0825	-0.6231	-0.4685	-0.4758	-0.4110	-0.4113	-0.4033	-0.4033
$b_{13,1}$	-3.4574	-0.1413	-1.3929	-3.2631	-3.0246	-3.4362	-3.4094	-3.4551	-3.4524
$b_{14,1}$	0.7728	1.1668	0.9919	0.7489	0.7497	0.7694	0.7695	0.7725	0.7725
$b_{15,1}$	-2.2208	0.1798	-0.8697	-2.0755	-1.8945	-2.2039	-2.1827	-2.2190	-2.2168
$b_{16,1}$	-0.6933	1.5151	1.5105	-0.4327	-0.2002	-0.6621	-0.6366	-0.6900	-0.6872
$b_{17,1}$	-0.6590	0.9249	0.4592	-0.6026	-0.5089	-0.6539	-0.6428	-0.6585	-0.6574
$b_{18,1}$	-2.1167	0.7813	-0.1834	-1.9790	-1.7981	-2.1027	-2.0817	-2.1153	-2.1131
$b_{19,1}$	-2.7036	0.4169	-0.2867	-2.4880	-2.2493	-2.6789	-2.6519	-2.7010	-2.6981
$b_{20,1}$	-3.7587	1.2185	0.5210	-3.2379	-2.7508	-3.6984	-3.6430	-3.7525	-3.7469
$b_{21,1}$	1.0717	1.9421	2.4297	1.2343	1.3627	1.0914	1.1041	1.0737	1.0751
$b_{22,1}$	1.0038	1.0916	1.1030	0.9292	0.9029	0.9941	0.9904	1.0028	1.0025
$b_{23,1}$	-1.4631	0.2710	-0.0951	-1.2857	-1.1442	-1.4406	-1.4238	-1.4607	-1.4590
$b_{24,1}$	**3.4800**	2.7546	3.4072	3.4861	3.4526	3.4813	3.4777	3.4801	3.4798
$b_{25,1}$	-2.9204	0.7603	-0.1304	-2.6721	-2.4025	-2.8932	-2.8617	-2.9176	-2.9145
$b_{1,2}$	**1.0900**	0.6202	1.3060	1.1906	1.2536	1.1005	1.1061	1.0911	1.0916
$b_{2,2}$	3.7520	1.0511	2.2124	3.6359	3.4836	3.7387	3.7207	3.7506	3.7488
$b_{3,2}$	4.1053	1.2131	2.5515	4.0249	3.8871	4.0971	4.0805	4.1045	4.1028
$b_{10,2}$	4.2658	1.0783	2.3000	4.1409	3.9571	4.2526	4.2316	4.2644	4.2622
$b_{13,2}$	3.3829	1.3048	2.6446	3.1891	3.0092	3.3602	3.3374	3.3805	3.3781
$b_{16,2}$	-5.8844	-1.3854	-2.9043	-5.7434	-5.5596	-5.8676	-5.8462	-5.8830	-5.8804
$b_{20,2}$	-5.0725	-1.4902	-3.0289	-5.0902	-4.9391	-5.0807	-5.0627	-5.0736	-5.0722
$b_{21,2}$	-7.6323	-1.8282	-3.8695	-7.4455	-7.2238	-7.6076	-7.5816	-7.6297	-7.6268
$b_{22,2}$	-2.3766	-0.5038	-1.0658	-2.2557	-2.1668	-2.3601	-2.3492	-2.3749	-2.3737
$b_{24,2}$	-9.1480	-2.1466	-4.5037	-8.9885	-8.7153	-9.1302	-9.0976	-9.1465	-9.1424
$b_{4,3}$	**0.8810**	0.1547	0.8806	0.8997	0.8810	0.8834	0.8810	0.8812	0.8810
$b_{5,3}$	0.9368	0.1598	0.9101	0.9445	0.9256	0.9382	0.9357	0.9369	0.9367
$b_{6,3}$	0.9068	0.1556	0.8858	0.9219	0.9000	0.9088	0.9061	0.9070	0.9067
$b_{7,4}$	0.4968	0.2286	0.4955	0.5005	0.4929	0.4971	0.4960	0.4968	0.4967
$b_{8,4}$	1.4797	0.6730	1.4523	1.5040	1.4782	1.4832	1.4803	1.4801	1.4798
$b_{9,4}$	**1.2300**	0.5678	1.2300	1.2487	1.2300	1.2324	1.2300	1.2302	1.2300
$b_{10,4}$	0.2988	0.2223	0.4561	0.3093	0.3111	0.3000	0.2999	0.2989	0.2989
$b_{11,4}$	0.6015	0.3961	0.8462	0.6526	0.6661	0.6082	0.6092	0.6022	0.6024
$b_{12,4}$	0.9443	0.5011	1.0804	0.9899	0.9903	0.9503	0.9501	0.9449	0.9449
$b_{15,4}$	1.0796	0.5301	1.1382	1.0917	1.0705	1.0814	1.0787	1.0798	1.0795
$b_{16,4}$	0.0517	0.0384	0.0689	0.0404	0.0304	0.0506	0.0494	0.0516	0.0515
$b_{18,4}$	0.3217	0.2107	0.4493	0.3348	0.3353	0.3234	0.3232	0.3218	0.3218
$b_{20,4}$	0.3486	0.2607	0.5137	0.3464	0.3238	0.3495	0.3467	0.3488	0.3485
$b_{13,5}$	**1.3900**	0.0079	0.0910	1.3583	1.2777	1.3879	1.3809	1.3898	1.3891
$b_{14,5}$	0.3644	0.0281	0.1816	0.3889	0.3901	0.3679	0.3680	0.3648	0.3648
$b_{15,5}$	1.6028	0.0490	0.3171	1.4567	1.3151	1.5855	1.5693	1.6010	1.5993
$b_{16,5}$	5.5055	0.2025	1.3085	5.1812	4.8481	5.4666	5.4293	5.5017	5.4975
$b_{17,5}$	1.4785	0.0739	0.4776	1.4275	1.3431	1.4739	1.4640	1.4780	1.4770
$b_{18,5}$	2.5876	0.1072	0.6919	2.4433	2.2766	2.5724	2.5536	2.5861	2.5841
$b_{19,5}$	2.8763	0.1119	0.7218	2.6807	2.4662	2.8539	2.8298	2.8740	2.8714
$b_{20,5}$	7.2672	0.2768	1.7649	6.8190	6.2974	7.2187	7.1593	7.2624	7.2565
$b_{21,5}$	5.3056	0.2000	1.3003	5.0216	4.7427	5.2696	5.2390	5.3019	5.2985
$b_{22,5}$	1.5715	0.0740	0.4829	1.5440	1.4983	1.5674	1.5624	1.5711	1.5705
$b_{23,5}$	1.5925	0.0579	0.3749	1.4313	1.3055	1.5719	1.5571	1.5903	1.5888
$b_{24,5}$	4.8563	0.2204	1.4148	4.7336	4.5597	4.8421	4.8211	4.8551	4.8524
$b_{25,5}$	3.4151	0.1421	0.9146	3.1890	2.9465	3.3903	3.3623	3.4126	3.4097

Table 4.14: Empirical result - Exact constraints vs Stochastic Case 3

Parameters	E	3A	3B	3C	3D	3E	3F	3G	3H
Error variance and covariance									
$\Theta_\epsilon(1,1)$	6.9486	6.8489	6.8406	6.9312	6.9233	6.9467	6.9462	6.9485	6.9484
$\Theta_\epsilon(2,2)$	5.2507	5.2276	5.2211	5.2489	5.2501	5.2505	5.2505	5.2507	5.2507
$\Theta_\epsilon(3,3)$	9.1033	8.8519	8.8624	9.0819	9.0582	9.1010	9.0991	9.1031	9.1028
$\Theta_\epsilon(4,4)$	5.8093	5.6845	5.6873	5.7720	5.7686	5.8059	5.8055	5.8090	5.8092
$\Theta_\epsilon(5,5)$	4.9639	5.0381	5.0361	4.9968	4.9924	4.9668	4.9665	4.9641	4.9639
$\Theta_\epsilon(6,6)$	4.2824	4.2317	4.2342	4.2596	4.2661	4.2807	4.2812	4.2823	4.2825
$\Theta_\epsilon(7,7)$	5.5819	4.5155	4.4721	5.3992	5.2696	5.5600	5.5487	5.5797	5.5785
$\Theta_\epsilon(8,8)$	6.6507	7.0232	7.0156	6.6793	6.7047	6.6537	6.6551	6.6510	6.6511
$\Theta_\epsilon(9,9)$	5.2989	5.4737	5.4671	5.3188	5.3284	5.3016	5.3026	5.2992	5.2994
$\Theta_\epsilon(10,10)$	14.0674	13.9011	13.9092	14.0386	14.0312	14.0633	14.0627	14.0670	14.0669
$\Theta_\epsilon(11,11)$	16.3770	16.1938	16.2099	16.3634	16.3589	16.3750	16.3748	16.3768	16.3765
$\Theta_\epsilon(12,12)$	12.1591	11.8441	11.8385	12.0943	12.0596	12.1503	12.1476	12.1581	12.1579
$\Theta_\epsilon(13,13)$	16.7540	16.5371	16.5742	16.7655	16.7832	16.7542	16.7563	16.7540	16.7545
$\Theta_\epsilon(14,14)$	7.5546	7.0348	6.9710	7.4333	7.3389	7.5404	7.5316	7.5532	7.5522
$\Theta_\epsilon(15,15)$	12.2074	12.3233	12.3110	12.2209	12.2308	12.2095	12.2104	12.2076	12.2078
$\Theta_\epsilon(16,16)$	25.7166	26.2521	26.2017	25.7429	25.7344	25.7213	25.7197	25.7164	25.7173
$\Theta_\epsilon(17,17)$	14.7645	14.5884	14.5998	14.7611	14.7706	14.7633	14.7641	14.7644	14.7645
$\Theta_\epsilon(18,18)$	28.0468	28.1072	28.1178	28.0792	28.0860	28.0503	28.0508	28.0472	28.0472
$\Theta_\epsilon(19,19)$	17.2525	17.1975	17.2052	17.2479	17.2540	17.2523	17.2525	17.2525	17.2530
$\Theta_\epsilon(20,20)$	28.8383	28.4515	28.5429	28.8291	28.9208	28.8305	28.8412	28.8380	28.8373
$\Theta_\epsilon(21,21)$	25.5322	26.0018	25.9299	25.5501	25.5262	25.5382	25.5342	25.5333	25.5325
$\Theta_\epsilon(22,22)$	23.4554	23.8115	23.8253	23.5409	23.5759	23.4667	23.4696	23.4566	23.4570
$\Theta_\epsilon(23,23)$	22.2282	22.1476	22.1449	22.2148	22.2316	22.2252	22.2272	22.2279	22.2282
$\Theta_\epsilon(24,24)$	61.5338	62.4378	62.4938	61.4840	61.4744	61.5261	61.5218	61.5330	61.5332
$\Theta_\epsilon(25,25)$	34.6262	34.2071	34.2629	34.6654	34.6914	34.6288	34.6329	34.6266	34.6262
$\Theta_\epsilon(2,1)$	1.4575	1.1433	1.1370	1.4164	1.3871	1.4532	1.4506	1.4571	1.4569
$\Theta_\epsilon(6,4)$	1.4927	1.3982	1.4009	1.4618	1.4627	1.4901	1.4900	1.4925	1.4927
$\Theta_\epsilon(7,6)$	0.3845	0.4254	0.4232	0.3890	0.3902	0.3851	0.3851	0.3846	0.3846
$\Theta_\epsilon(8,5)$	0.0283	-0.0346	-0.0309	0.0196	0.0156	0.0269	0.0265	0.0281	0.0280
$\Theta_\epsilon(11,5)$	2.3114	2.2930	2.2933	2.3105	2.3083	2.3113	2.3111	2.3114	2.3113
$\Theta_\epsilon(11,8)$	-0.6303	-1.0117	-0.9937	-0.6978	-0.7328	-0.6397	-0.6431	-0.6314	-0.6320
$\Theta_\epsilon(13,10)$	8.4350	8.2002	8.2258	8.4323	8.4396	8.4339	8.4350	8.4349	8.4351
$\Theta_\epsilon(14,7)$	3.1800	2.3878	2.3346	3.0230	2.9105	3.1611	3.1509	3.1780	3.1769
$\Theta_\epsilon(18,17)$	1.8727	1.9487	1.9376	1.8539	1.8481	1.8707	1.8697	1.8725	1.8726
$\Theta_\epsilon(19,15)$	-0.6215	-0.7301	-0.7189	-0.5953	-0.5697	-0.6188	-0.6159	-0.6212	-0.6209
$\Theta_\epsilon(19,17)$	4.4152	4.5256	4.5308	4.4373	4.4662	4.4170	4.4197	4.4153	4.4154
$\Theta_\epsilon(25,18)$	8.5309	8.4177	8.4383	8.5507	8.5489	8.5324	8.5327	8.5312	8.5312

Chapter 5

Simulation study

In Chapter 3, we have developed the theoretical details of incorporating stochastic constraints into the SEM analysis of APT. In order to examine the behaviour of the parameter estimates with stochastic constraints, a simulation study has been conducted. Comparison for the parameter estimates with exact and stochastic constraints have been made in different designs.

5.1 Simulation design

In the simulation study, we used a model with eight observed variables and three latent factors f_1 f_2 and f_3. f_1 is considered to be the general factor for all variables while f_2 and f_3 are the specific factors of the first four and the last four variables respectively. The relationship between security return r_i and the factors is shown by the following equation:

$$r_i = r_f + b_{i1}(\lambda_1 + f_1) + b_{i2}(\lambda_2 + f_2) + b_{i3}(\lambda_3 + f_3) + \epsilon_i \tag{5.1}$$

for $i = 1, 2, \cdots, 8$, where r_f is the risk free rate, b_{ij} is the factor loading of the f_j to r_i and λ_j is the risk premia with j = 1, 2, 3. The study is based on the confirmatory factor analysis model with the covariance of the observed variables given by:

$$\mathbf{\Sigma} = \mathbf{B\Phi B}' + \mathbf{\Theta}_\epsilon$$

where $\mathbf{B}$ is a 8x3 factor loading matrix, $\mathbf{\Phi}$ is a 3x3 correlation matrix for the factors and $\mathbf{\Theta}_\epsilon$ is a 8x8 error covariance matrix. The mean of the observed variables is given by:

$$\boldsymbol{\mu} = r_f\mathbf{1} + \mathbf{B\Lambda}$$

where $\mathbf{1}$ is a 8x1 vector with all elements equal to 1 and $\mathbf{\Lambda} = (\lambda_1, \lambda_2, \lambda_3)^T$ is a 3x1 risk premia mean vector.

The simulation is conducted for different combinations of the sample size n, the true parameter values of the factor loading matrix $\mathbf{B}$, factor correlation matrix $\mathbf{\Phi}$ and the risk premia vector $\mathbf{\Lambda}$. First of all, we consider two sample sizes, namely $n = 2000$ and 200, with a veiw to compare the effect of sample sizes to the estimation results. After that, two sets of factor loading matrices are constructed and they are used to examine the estimation performance between the large and small factor loadings. They are differ by the true values of the parameters corresponding to the general factors and they are given as follows:

$$\mathbf{B1} = \begin{pmatrix} b_{11} & b_{12}(ref) & \mathbf{b_{13}} \\ b_{21} & b_{22} & \mathbf{b_{23}} \\ b_{31} & b_{32} & \mathbf{b_{33}} \\ b_{41} & b_{42} & \mathbf{b_{43}} \\ b_{51} & \mathbf{b_{52}} & b_{53}(ref) \\ b_{61} & \mathbf{b_{62}} & b_{63} \\ b_{71} & \mathbf{b_{72}} & b_{73} \\ b_{81}(ref) & \mathbf{b_{82}} & b_{83} \end{pmatrix} = \begin{pmatrix} 0.65 & 1.0(ref) & \mathbf{0.0} \\ 0.70 & 0.6 & \mathbf{0.0} \\ 0.75 & 0.7 & \mathbf{0.0} \\ 0.80 & 0.8 & \mathbf{0.0} \\ 0.85 & \mathbf{0.0} & 1.0(ref) \\ 0.90 & \mathbf{0.0} & 0.2 \\ 0.95 & \mathbf{0.0} & 0.3 \\ 1.00(ref) & \mathbf{0.0} & 0.4 \end{pmatrix}$$

and

$$\mathbf{B2} = \begin{pmatrix} b_{11} & b_{12}(ref) & \mathbf{b_{13}} \\ b_{21} & b_{22} & \mathbf{b_{23}} \\ b_{31} & b_{32} & \mathbf{b_{33}} \\ b_{41} & b_{42} & \mathbf{b_{43}} \\ b_{51} & \mathbf{b_{52}} & b_{53}(ref) \\ b_{61} & \mathbf{b_{62}} & b_{63} \\ b_{71} & \mathbf{b_{72}} & b_{73} \\ b_{81}(ref) & \mathbf{b_{82}} & b_{83} \end{pmatrix} = \begin{pmatrix} 0.30 & 1.0(ref) & \mathbf{0.0} \\ 0.35 & 0.6 & \mathbf{0.0} \\ 0.40 & 0.7 & \mathbf{0.0} \\ 0.45 & 0.8 & \mathbf{0.0} \\ 0.50 & \mathbf{0.0} & 1.0(ref) \\ 0.55 & \mathbf{0.0} & 0.2 \\ 0.60 & \mathbf{0.0} & 0.3 \\ 0.65(ref) & \mathbf{0.0} & 0.4 \end{pmatrix}$$

Note that the values in boldface are fixed. In both settings, the observed variables r_8, r_1 and r_5 are selected to be the reference variables for latent factors f_1, f_2 and f_3 respectively.

Regarding the factor correlation matrices, the two specific factors are set to be uncorrelated between each other in all designs. However, different correlations between the general and the specific factors are considered in the following two matrices. They are:

$$\mathbf{\Phi}1 = \begin{pmatrix} \mathbf{1.0} & 0.5 & 0.5 \\ 0.5 & \mathbf{1.0} & 0.0 \\ 0.5 & 0.0 & \mathbf{1.0} \end{pmatrix} \text{ and } \mathbf{\Phi}2 = \begin{pmatrix} \mathbf{1.0} & 0.0 & 0.0 \\ 0.0 & \mathbf{1.0} & 0.0 \\ 0.0 & 0.0 & \mathbf{1.0} \end{pmatrix}$$

Concerning the risk premia vectors, the two sets that we studied are $\mathbf{\Lambda}1 = (\lambda_1{=}1, \lambda_2{=}2, \lambda_3{=}3)$ and $\mathbf{\Lambda}2 = (\lambda_1{=}1, \lambda_2{=}2, \lambda_3{=}0)$. They are set to assess whether the estimation performs well when one of the risk permium equals to zero. For simplicity, the risk free rate r_f is taken to be equal to 0.5 and the error terms are considered to be uncorrelated among each other and with variances equal to 0.5 in all designs. In total, there are 16 designs and Table 5.1 shows the summary of the simulation designs.

Table 5.1: Summary of simulation designs

Design	Sample Size	Factor loadings	Factor correlation	Risk Premia			
				$\mathbf{\Lambda}$:	λ_1	λ_2	λ_3
A	2000, 200	**B1**	$\mathbf{\Phi}1$	$\mathbf{\Lambda}1$:	1	2	3
B	2000, 200	**B1**	$\mathbf{\Phi}1$	$\mathbf{\Lambda}2$:	1	2	0
C	2000, 200	**B1**	$\mathbf{\Phi}2$	$\mathbf{\Lambda}1$:	1	2	3
D	2000, 200	B1	$\Phi 2$	$\Lambda 2$:	1	2	0
E	2000, 200	B2	$\Phi 1$	$\Lambda 1$:	1	2	3
F	2000, 200	B2	$\Phi 1$	$\Lambda 2$:	1	2	0
G	2000, 200	B2	$\Phi 2$	$\Lambda 1$:	1	2	3
H	2000, 200	B2	$\Phi 2$	$\Lambda 2$:	1	2	0

In this study, we are mainly interested in setting the constraints on the risk free rate (r_f) and reference loadings (b_{81}, b_{12} and b_{53}). As our aim is to compare the estimations with the use of exact constraints and different cases of stochastic constraints, all 16 designs are analysed by nine methods. Table 5.2 and 5.3 show the form of constraints under different factor loading settings and the nine analysing methods respectively.

Table 5.2: Constraints

Factor loading	Exact constraints	Stochastic constraints
B1	$r_f = 0.50$	$0.50 = r_f + \epsilon_{u1}$
	$b_{81} = 1.00$	$1.00 = b_{81} + \epsilon_{u2}$
	$b_{12} = 1.00$	$1.00 = b_{12} + \epsilon_{u3}$
	$b_{53} = 1.00$	$1.00 = b_{53} + \epsilon_{u4}$
B2	$r_f = 0.50$	$0.50 = r_f + \epsilon_{u1}$
	$b_{81} = 0.65$	$0.65 = b_{81} + \epsilon_{u2}$
	$b_{12} = 1.00$	$1.00 = b_{12} + \epsilon_{u3}$
	$b_{53} = 1.00$	$1.00 = b_{53} + \epsilon_{u4}$

Table 5.3: Summary of methods

Method	**Constraints**	**Values of prior constants**
Exact	Exact	N.A.
1a	Stochastic (Case 1)	$\nu = 5$, $\beta = 0.1$
1b	Stochastic (Case 1)	$\nu = 500$, $\beta = 0.01$
2a	Stochastic (Case 2)	$\nu_1 = 5$, $\beta_1 = 0.1$ for r_f $\nu_2 = 5$, $\beta_2 = 0.01$ for b_{81}, b_{12} and b_{53}
2b	Stochastic (Case 2)	$\nu_1 = 500$, $\beta_1 = 0.01$ for r_f $\nu_2 = 500$, $\beta_2 = 0.001$ for b_{81}, b_{12} and b_{53}
2c	Stochastic (Case 2)	$\nu_1 = 5$, $\beta_1 = 0.01$ for r_f $\nu_2 = 5$, $\beta_2 = 0.1$ for b_{81}, b_{12} and b_{53}
3a	Stochastic (Case 3)	$\rho = 5$, matrix **R**=**R**1
3b	Stochastic (Case 3)	$\rho = 500$, matrix **R**=**R**1
3aR	Stochastic (Case 3)	$\rho = 5$, matrix **R**=**R2**

$$\text{where } \mathbf{R1} = \begin{pmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 1.0 \end{pmatrix} \text{ and } \mathbf{R2} = \begin{pmatrix} 1.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 1.0 & 0.5 & 0.5 \\ 0.0 & 0.5 & 1.0 & 0.5 \\ 0.0 & 0.5 & 0.5 & 1.0 \end{pmatrix}$$

These matrices are selected to investigate whether the estimation result will be affected by the dependency among the errors for the stochastic constraints. For matrix **R**1, it assumes that the errors are independent among each other. For matrix **R**2, it assumes that the errors for the stochastic constraints on the reference loadings depend on each other but they are independent of that on the risk free rate r_f.

5.2 Simulation procedure

For each combination, 100 replications are generated. The simulation is conducted by the steps listed below:

1. Simulate a sample of size n of latent factors (f_1, f_2 and f_3) from $N_3(0, \boldsymbol{\Phi i})$ (where $i = 1,2$) and the random errors from $N_8(0, \Theta_\epsilon)$ by S-plus.
2. Calculate the eight observed variables (r_1 to r_8) by substituting the assigned true parameters value together with the factors and errors generated from step 1 into equation (5.1).
3. Mx is employed to analyse the simulated observed variates.

During the analysis, we have to consider the starting values of the unknown parameters. For the cases with exact constraints, the estimation result is unaffected by the starting values. The estimates of the unknown parameters remain the same for different sets of starting value. However, for those cases with stochastic constraints, some replications do not converge and some give unreasonable estimates with arbitrary staring values generated by the Mx. Moreover, the estimation results are not stable with different starting values. To achieve the optimum estimation result for the cases with stochastic constraints, we use the estimates obtained from the corresponding cases with exact constraints as the starting values of the unknown parameters.

After that, to study the accuracy of the estimation, the following statistics of all parameter estimates are computed for every combination. Let $\boldsymbol{\theta}$ and $\hat{\boldsymbol{\theta}}_{\boldsymbol{i}}$ be the parameter vectors with the true assigned value and the estimated value obtained in the i^{th} replication respectively, we have computed the following statistics of the sample estimates:

Sample mean of the estimated parameter vector:

$$\bar{\hat{\boldsymbol{\theta}}} = \frac{1}{100}\sum_{i=1}^{100}\hat{\boldsymbol{\theta}}_{\boldsymbol{i}}$$

Bias:

$$\text{Bias} = \bar{\hat{\boldsymbol{\theta}}} - \boldsymbol{\theta}$$

Let $\boldsymbol{\theta_j}$ and $\hat{\boldsymbol{\theta_{ij}}}$ be the j^{th} elements of $\boldsymbol{\theta}$ and $\hat{\boldsymbol{\theta}}_i$ respectively, the root mean square error (RMSE)of the j^{th} element is given by:

$$\text{RMSE} = \sqrt{\frac{1}{100}\sum_{i=1}^{100}(\hat{\boldsymbol{\theta_{ij}}} - \boldsymbol{\theta_j})^2}$$

These statistics are presented in Table A1 - A8, Table B1 - B8, and Table C1 - C8 respectively. For the purpose of comparing the effects of the analysis methods (the stochastic constraints refer) and the sample sizes, each table contains the result of the combinations which are only differ by the analysis methods and the sample size. For some designs, we encounter a few non-converged replications. The result of the non-converged cases are excluded when the aforementioned statistics are computed. The number of converged cases of each combination is reported for reference.

5.3 Simulation result

From the calculated statistics in the simulation study, we have the following observations.

5.3.1 Sample size

When comparing the results of the designs with sample size 2000 and those with 200, it is found that as the sample size decreases, the bias and RMSE of the estimates increase. Please refer to Appendices B and C for details. The accuracy of the estimation decreases with the sample size of the data is reasonable.

5.3.2 Analysis methods (constraints)

Exact vs. Stochastic

When $n = 2000$, most of the RMSE corresponding to the estimates from exact constraint models are slightly smaller than those obtained from stochastic constraint models. However, when the sample size decreases to 200, the RMSE of the estimates obtained from models with exact constraints are similar to those with stochastic constraints. Please refer to Appendix C for details. This indicates that the estimates produced with stochastic constraints are comparable to those produced with exact constraints.

Case 1 vs. Case 2 vs. Case 3

When comparing the stochastic constraint models with different cases (Case 1, 2 and 3). It is found that for both $n = 2000$ and $n = 200$, most of the estimates are similar and their bias and RMSE are close to each other.

Case 1 and Case 2

For models with the Case 1 and Case 2 stochastic constraints, we consider several cases with different values of ν and β. It is observed that when we increase the value of ν and decrease the value of β, the estimates will get closer to the results of the exact constraint models. Moreover, most of the bias and RMSE will become smaller and the effect is more noticeable for those variables with stochastic constraints (r_f, b_{81}, b_{12} and b_{53}).

Case 3

Similarly, we also consider several cases of stochastic constraint models using Case 3 by changing the value of ρ and the matrix $\mathbf{R}$. When the ρ's value is increased (Case 3b refers), the estimates approaches to those achieved by the exact constraints models. Moreover, this reduces most of the bias and RMSE of the estimates. In addition, we get similar result from combinations using analysis methods **3a** and **3aR**. Since both methods use stochastic constraints Case 3 with $\rho = 5$ but different **R** matrices, the similarity indicates that the estimation results are not affected significantly by the form of matrix **R**.

5.3.3 Factor loadings

In order to examine the estimation performance between the large and small factor loading, we have compared the simulation results of design A with E, design B with F, design C with G, and design D with H. It is found that the bias and the RMSE of the estimates obtained in designs A and C are similar to those of designs E and G respectively. Although some estimates in design B and D have different bias and RMSE values from those of designs F and H, there is no clear pattern showing which design performs better than the other. Therefore, we may conclude that there is no significant difference between the estimation of APT model with larger and smaller general factor loadings.

5.3.4 Factor correlation matrix

To examine whether the existence of correlations between the general and specific factors affects the estimation result, we have compared the simulation results of design A with C, design B with D, design E with G, and design F with H. In the design comparison, we observed that some differences exist in the bias and RMSE of several estimates. However, there is no obvious pattern showing which design is superior to the other. Hence, no evidence shows that the existence of correlations between the general and specific factors affects the estimation result.

5.3.5 Risk premia

To check whether the estimation performs well when one of the risk premium equals to zero, we have compared the estimation result of design A with B, design C with D, design E with F, and design G with H. Here are the observations. For designs with $n = 2000$, most estimates of factor loading of two specific factors f_2

and f_3, and the correlations of the latent factors in designs with $\lambda_3 = 0$ have larger bias and RMSE than those with $\lambda_3 = 3$. This phenomenon is more significant in design A with B and design E with F. However, this pattern is less obvious in designs with $n = 200$. In general, there is not enough evidence to conclude that the estimation of APT model with non-zero risk premium perform better than that with one of the risk premium equals to zero.

5.3.6 Overall result

When comparing the simulation results of the designs with different factor loading matrices (B1 and B2), factor correlation matrices ($\mathbf{\Phi}1$ and $\mathbf{\Phi}2$) and risk premia vectors ($\mathbf{\Lambda}1$ and $\mathbf{\Lambda}2$), it found that the estimation accuracy of the model may be affected by the form of its components ($\mathbf{B}$, $\mathbf{\Phi}$, and $\mathbf{\Lambda}$). However, in general, we can see that the bias and the RMSE of most of the estimates in different simulation designs are small, and the mean values are close to the true parameters values (please refer to Appendix A). It shows that the estimation of the SEM analysis of APT with stochastic constraints perform well under different settings.

Chapter 6

Conclusion and discussion

In this thesis, we have further developed the SEM analysis of APT model by imposing stochastic constraints to fixed parameters for the purpose of the identification of the model or incorporation of prior information into the analysis. If we are completely sure of the validity of the prior information, we should use exact constraints. Otherwise, the more flexible stochastic constraints are advisable because they allow some randomness exist between the prior information and the fixed parameters. Hence, it provides more meaningful interpretation of the model.

The major objective of the thesis is to develop the APT models with three types of stochastic constraints. It is found that each proposed model consists of two components, the Maximum Likelihood fit function with mean structure and a user defined fit function. In practical applications, analysts can conveniently implement the proposed model using the Mx software program. However, it is worthy of note that Mx does not provide the standard error parameter estimates and the accuracy of the parameter estimates are reflected by the confidence intervals. As the confidence intervals involves very heavy computation, the thesis has not reported confidence intervals. In order to have a complete investigation of the proposed model, we have conducted an empirical study and a simulation study.

The empirical study based on Hong Kong Stock Market data demonstrates the performance of the proposed model in realistic situation. The analysis is founded on a good fit confirmatory factor analysis APT model and we are interested in how the estimation results are affected by the prior constants' value. In general, we choose these values with reference to degree of certainly on the prior information. From the results of the study, it is found, as expected, that when the prior distribution are set to be more informative, the differences between the estimation results obtained from models with stochastic constraints and those with exact constraints diminish. It is worthy of note that if prior constants were set to reflect non-information prior distributions, then unreasonable estimates might be given.

In the simulation study, we analyse the simulated data through the variations in different components of the model. Although there exists some difference in the precision of the estimation under different designs, the bias and RMSE of most of the estimates are small. In general, the proposed model produces accurate estimates in various settings of the model, including the use of different cases of stochastic constraints.

Both the empirical and simulation studies indicate that it is sensible to incorporate stochastic constraints into the SEM analysis of APT model for practical applications.

Appendix A

Simulation result - Mean

Table A.1: Simulation result - Mean

Design A	Factor loadings = B1			Factor correlation = $\Phi 1$			$\lambda_1 = 1$ $\lambda_2 = 2$ $\lambda_3 = 3$		
Constraints	Exact	1a	1b	2a	2b	2c	3a	3b	3aR
				$n = 2000$					
No of cases converaged	98	97	100	94	100	92	95	95	99
True value of parameters									
b_{11}=0.65	0.6484	0.6435	0.6444	0.6452	0.6469	0.6425	0.6429	0.6452	0.6468
b_{21}=0.70	0.7101	0.6956	0.6963	0.6968	0.6983	0.6944	0.6945	0.6961	0.6980
b_{31}=0.75	0.7528	0.7474	0.7481	0.7496	0.7504	0.7453	0.7468	0.7489	0.7498
b_{41}=0.80	0.8045	0.7941	0.7943	0.7957	0.7969	0.7920	0.7934	0.7950	0.7968
b_{51}=0.85	0.8426	0.8380	0.8405	0.8395	0.8474	0.8376	0.8344	0.8474	0.8441
b_{61}=0.90	0.8880	0.8896	0.8911	0.8910	0.8941	0.8891	0.8890	0.8934	0.8908
b_{71}=0.95	0.9451	0.9448	0.9459	0.9458	0.9495	0.9433	0.9430	0.9494	0.9455
b_{81}=1.00	**1.0000**	0.9921	0.9937	0.9947	0.9984	0.9915	0.9904	0.9974	0.9943
b_{12}=1.00	**1.0000**	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9972
b_{22}=0.60	0.5911	0.6003	0.6000	0.5997	0.5996	0.6010	0.6009	0.5999	0.5985
b_{32}=0.70	0.6946	0.6971	0.6971	0.6955	0.6966	0.6980	0.6976	0.6968	0.6952
b_{42}=0.80	0.7929	0.7994	0.7998	0.7981	0.7993	0.8008	0.8000	0.7993	0.7973
b_{53}=1.00	**1.0000**	1.0001	0.9990	0.9990	0.9993	0.9998	0.9990	0.9990	0.9974
b_{63}=0.20	0.1991	0.2017	0.2012	0.2000	0.2006	0.2027	0.2017	0.2007	0.2016
b_{73}=0.30	0.2985	0.2996	0.2991	0.2977	0.2986	0.3008	0.2996	0.2983	0.2993
b_{83}=0.40	0.3952	0.4002	0.3998	0.3981	0.3991	0.4016	0.4001	0.3993	0.3995
r_f=0.50	**0.5000**	0.4931	0.4991	0.4897	0.4982	0.5008	0.4962	0.4997	0.4944
λ_1=1.00	1.0169	1.0029	0.9983	1.0134	1.0011	0.9913	0.9978	1.0002	0.9996
λ_2=2.00	1.9982	2.0096	2.0052	2.0054	2.0020	2.0083	2.0079	2.0033	2.0143
λ_3=3.00	2.9916	3.0184	3.0163	3.0127	3.0050	3.0206	3.0273	3.0051	3.0234
ϕ_{21}=0.50	0.4963	0.5027	0.5014	0.5002	0.4976	0.5040	0.5033	0.4996	0.5009
ϕ_{31}=0.50	0.5118	0.5212	0.5161	0.5164	0.5040	0.5190	0.5275	0.5051	0.5160
ϕ_{32}=0.00	-0.0081	-0.0009	-0.0020	-0.0040	-0.0048	0.0011	0.0007	-0.0045	-0.0118
				$n = 200$					
No of cases converaged	97	96	98	98	98	99	97	98	98
True values of parameters									
b_{11}=0.65	0.6625	0.6773	0.6719	0.6805	0.6732	0.7101	0.6831	0.6657	0.6613
b_{21}=0.70	0.7084	0.7218	0.7186	0.7506	0.7155	0.7428	0.7255	0.7117	0.7172
b_{31}=0.75	0.7503	0.7628	0.7611	0.8075	0.7593	0.7895	0.7693	0.7533	0.7598
b_{41}=0.80	0.7963	0.8077	0.8071	0.9349	0.8047	0.8393	0.8131	0.7987	0.8032
b_{51}=0.85	0.8231	0.7966	0.8171	0.8487	0.8263	0.8257	0.7861	0.8247	0.8855
b_{61}=0.90	0.8983	0.8987	0.9009	0.9523	0.8992	0.9014	0.8878	0.8995	0.8989
b_{71}=0.95	0.9530	0.9462	0.9541	0.9401	0.9542	0.9537	0.9416	0.9543	0.9617
b_{81}=1.00	**1.0000**	0.9887	0.9987	1.0187	1.0002	0.9993	0.9866	1.0002	1.0131
b_{12}=1.00	**1.0000**	1.0082	1.0045	1.0000	1.0008	1.0093	1.0122	1.0015	1.0277
b_{22}=0.60	0.6018	0.6004	0.5999	0.5872	0.6017	0.6016	0.5999	0.6016	0.6089
b_{32}=0.70	0.7052	0.7047	0.7032	0.7364	0.7049	0.7052	0.7031	0.7054	0.7138
b_{42}=0.80	0.8007	0.8034	0.8001	0.7426	0.8017	0.8034	0.8019	0.8019	0.8134
b_{53}=1.00	**1.0000**	1.0293	1.0062	1.0147	1.0006	1.0329	1.0646	1.0012	1.0568
b_{63}=0.20	0.2092	0.2006	0.2030	0.1832	0.2083	0.2045	0.2151	0.2083	0.2104
b_{73}=0.30	0.3108	0.3073	0.3054	0.2703	0.3095	0.3105	0.3188	0.3101	0.3163
b_{83}=0.40	0.4056	0.4062	0.4030	0.3917	0.4056	0.4101	0.4175	0.4056	0.4147
r_f=0.50	**0.5000**	0.4936	0.4993	0.4963	0.4993	0.4986	0.4651	0.4999	0.4692
λ_1=1.00	0.8812	0.9583	0.9279	0.8670	0.8873	0.9443	0.9866	0.8861	0.9823
λ_2=2.00	2.0205	1.9577	1.9866	1.7417	2.0026	1.9491	1.9538	2.0123	1.9733
λ_3=3.00	3.1778	3.0693	3.1157	2.7050	3.1695	3.0352	3.0200	3.1673	2.9498
ϕ_{21}=0.50	0.4900	0.4728	0.4722	0.4334	0.4770	0.4334	0.4599	0.4845	0.4770
ϕ_{31}=0.50	0.5214	0.5650	0.5297	0.4898	0.5185	0.5198	0.5605	0.5182	0.4898
ϕ_{32}=0.00	-0.0294	0.0040	-0.0152	0.1434	-0.0457	-0.0142	0.0301	-0.0277	-0.0500

Remarks: The values in boldface are fixed

Table A.2: Simulation result - Mean

Design B	Factor loadings = B1		Factor correlation = $\Phi 1$			$\lambda_1 = 1$ $\lambda_2 = 2$ $\lambda_3 = 0$			
Constraints	Exact	1a	1b	2a	2b	2c	3a	3b	3aR
				$n = 2000$					
No of cases converaged	100	100	100	100	100	100	100	100	100
True value of parameters									
b_{11}=0.65	0.6505	0.6751	0.6492	0.6566	0.6377	0.6546	0.6805	0.6544	0.6074
b_{21}=0.70	0.7177	0.7665	0.7137	0.7372	0.7075	0.7291	0.7714	0.7195	0.7310
b_{31}=0.75	0.7615	0.8138	0.7662	0.7840	0.7582	0.7813	0.8189	0.7638	0.7706
b_{41}=0.80	0.8121	0.8673	0.8113	0.8350	0.8024	0.8279	0.8728	0.8148	0.8173
b_{51}=0.85	0.8224	0.8645	0.8523	0.8423	0.8451	0.8749	0.8777	0.8252	0.8770
b_{61}=0.90	0.9076	0.9248	0.9142	0.9101	0.9049	0.9216	0.9302	0.9110	0.9348
b_{71}=0.95	0.9585	0.9830	0.9648	0.9666	0.9549	0.9742	0.9908	0.9622	0.9956
b_{81}=1.00	**1.0000**	1.0317	1.0130	1.0129	1.0027	1.0251	1.0412	1.0042	1.0459
b_{12}=1.00	**1.0000**	1.0005	1.0000	1.0000	1.0000	1.0002	1.0006	1.0000	1.0312
b_{22}=0.60	0.5855	0.5449	0.5859	0.5648	0.5838	0.5743	0.5430	0.5863	0.5580
b_{32}=0.70	0.6884	0.6460	0.6833	0.6645	0.6818	0.6725	0.6449	0.6892	0.6633
b_{42}=0.80	0.7875	0.7470	0.7877	0.7661	0.7860	0.7763	0.7465	0.7884	0.7679
b_{53}=1.00	**1.0000**	1.0315	1.0116	1.0144	1.0023	1.0211	1.0385	1.0033	1.0469
b_{63}=0.20	0.1822	0.1829	0.1921	0.1916	0.2024	0.1896	0.1773	0.1787	0.1765
b_{73}=0.30	0.2917	0.2896	0.2966	0.2986	0.3072	0.2930	0.2821	0.2880	0.2818
b_{83}=0.40	0.3972	0.3898	0.3968	0.4008	0.4082	0.3912	0.3806	0.3930	0.3810
r_f=0.50	**0.5000**	0.5308	0.5013	0.5296	0.4994	0.5021	0.5450	0.5019	0.5459
λ_1=1.00	1.0242	1.0129	1.0201	1.0133	1.0339	1.0245	0.9946	1.0181	0.9935
λ_2=2.00	1.9997	1.9558	1.9951	1.9723	1.9995	1.9903	1.9490	1.9979	1.9595
λ_3=0.00	-0.0092	-0.0730	-0.0341	-0.0552	-0.0396	-0.0582	-0.0794	-0.0083	-0.0867
ϕ_{21}=0.50	0.4948	0.4734	0.4994	0.4945	0.5177	0.4956	0.4659	0.4887	0.5222
ϕ_{31}=0.50	0.5324	0.4653	0.4884	0.4953	0.4961	0.4611	0.4515	0.5288	0.4479
ϕ_{32}=0.00	-0.0231	-0.0497	-0.0268	-0.0355	-0.0248	-0.0378	-0.0545	-0.0244	-0.0235
				$n = 200$					
No of cases converaged	100	100	98	99	99	98	95	100	100
True values of parameters									
b_{11}=0.65	0.6677	0.6838	0.6911	0.7496	0.6635	0.7391	0.6576	0.7050	0.6399
b_{21}=0.70	0.7638	0.7644	0.7882	0.8100	0.7691	0.8319	0.7934	0.7925	0.7504
b_{31}=0.75	0.8007	0.7768	0.8266	0.8543	0.8078	0.8723	0.8212	0.8304	0.7796
b_{41}=0.80	0.8309	0.8238	0.8608	0.9004	0.8424	0.9091	0.8685	0.8681	0.8142
b_{51}=0.85	0.7947	0.7488	0.8374	0.8188	0.8010	0.8938	0.9133	0.7892	0.9389
b_{61}=0.90	0.9045	0.9247	0.9085	0.9112	0.9031	0.9316	0.8878	0.9074	0.9269
b_{71}=0.95	0.9604	0.9832	0.9648	0.9720	0.9584	0.9942	0.9628	0.9614	0.9960
b_{81}=1.00	**1.0000**	1.0095	1.0068	1.0041	1.0009	1.0272	1.0089	1.0016	1.0497
b_{12}=1.00	**1.0000**	1.0038	1.0036	1.0029	1.0005	1.0152	1.0334	1.0010	1.0529
b_{22}=0.60	0.5305	0.5413	0.5155	0.5206	0.5238	0.5118	0.4999	0.5111	0.5576
b_{32}=0.70	0.6370	0.6756	0.6222	0.6265	0.6293	0.6216	0.6190	0.6180	0.6755
b_{42}=0.80	0.7527	0.7855	0.7350	0.7312	0.7403	0.7366	0.7210	0.7277	0.7877
b_{53}=1.00	**1.0000**	1.0399	1.0176	1.0122	1.0021	1.0534	1.0660	1.0040	1.0861
b_{63}=0.20	0.2325	0.2082	0.2400	0.2171	0.2355	0.2215	0.2913	0.2295	0.2067
b_{73}=0.30	0.3342	0.2975	0.3444	0.3132	0.3381	0.3151	0.3814	0.3301	0.2982
b_{83}=0.40	0.4294	0.3789	0.4398	0.4203	0.4321	0.4224	0.4806	0.4285	0.3908
r_f=0.50	**0.5000**	0.5538	0.4994	0.5008	0.5005	0.4993	0.4961	0.5000	0.5376
λ_1=1.00	1.1124	0.9772	1.1227	1.0969	1.1184	1.1056	1.0864	1.1115	1.0344
λ_2=2.00	1.9065	1.9200	1.8660	1.8001	1.9092	1.8271	1.8167	1.8500	1.8462
λ_3=0.00	-0.0927	-0.0127	-0.1361	-0.0963	-0.1066	-0.1615	-0.1336	-0.0843	-0.1547
ϕ_{21}=0.50	0.5056	0.4762	0.4828	0.4117	0.5125	0.4197	0.4725	0.4647	0.5111
ϕ_{31}=0.50	0.5219	0.6326	0.4609	0.5032	0.5114	0.3994	0.3852	0.5280	0.4051
ϕ_{32}=0.00	-0.0439	0.0034	-0.0453	-0.0409	-0.0471	-0.0653	0.0115	-0.0405	-0.0575

Remarks: The values in boldface are fixed

Table A.3: Simulation result - Mean

Design C	Factor loadings = B1			Factor correlation = $\Phi 2$			$\lambda_1 = 1$ $\lambda_2 = 2$ $\lambda_3 = 3$		
Constraints	Exact	1a	1b	2a	2b	2c	3a	3b	3aR
				$n = 2000$					
No of cases converaged	100	100	100	92	99	100	100	100	100
True value of parameters									
b_{11}=0.65	0.7587	0.7563	0.6728	0.7462	0.6595	0.7638	0.7573	0.6494	0.7677
b_{21}=0.70	0.7671	0.7654	0.7165	0.7645	0.7080	0.7702	0.7664	0.7059	0.7721
b_{31}=0.75	0.8257	0.8248	0.7668	0.8224	0.7569	0.8300	0.8255	0.7510	0.8326
b_{41}=0.80	0.8864	0.8859	0.8194	0.8800	0.8079	0.8915	0.8865	0.8035	0.8951
b_{51}=0.85	0.8608	0.8362	0.8439	0.8580	0.8593	0.8434	0.8417	0.8616	0.8392
b_{61}=0.90	0.8972	0.8931	0.8950	0.8860	0.8967	0.8941	0.8943	0.8871	0.8935
b_{71}=0.95	0.9502	0.9435	0.9461	0.9471	0.9500	0.9453	0.9453	0.9483	0.9440
b_{81}=1.00	**1.0000**	0.9909	0.9946	0.9992	0.9994	0.9931	0.9931	1.0002	0.9917
b_{12}=1.00	**1.0000**	1.0064	1.0087	1.0085	1.0055	1.0076	1.0091	1.0064	1.0092
b_{22}=0.60	0.6017	0.6029	0.6050	0.6051	0.6036	0.6038	0.6047	0.6049	0.6043
b_{32}=0.70	0.7024	0.7044	0.7062	0.7067	0.7047	0.7051	0.7062	0.7072	0.7066
b_{42}=0.80	0.8026	0.8052	0.8069	0.8070	0.8054	0.8059	0.8071	0.8051	0.8080
b_{53}=1.00	**1.0000**	1.0015	1.0003	0.9969	0.9998	1.0008	1.0009	0.9979	1.0041
b_{63}=0.20	0.2099	0.2022	0.2023	0.2095	0.2032	0.2022	0.2021	0.2113	0.2026
b_{73}=0.30	0.3086	0.3019	0.3018	0.3076	0.3025	0.3017	0.3016	0.3092	0.3027
b_{83}=0.40	0.4096	0.4031	0.4025	0.4055	0.4035	0.4028	0.4027	0.4075	0.4041
r_f=0.50	**0.5000**	0.5115	0.5019	0.5108	0.5025	0.5017	0.5034	0.5003	0.5186
λ_1=1.00	0.9794	0.9818	0.9899	0.9648	0.9858	0.9919	0.9902	0.9696	0.9743
λ_2=2.00	1.9066	1.8844	1.9690	1.9006	1.9903	1.8783	1.8804	2.0095	1.8651
λ_3=3.00	3.0063	3.0268	3.0184	3.0205	3.0009	3.0195	3.0201	3.0206	3.0160
ϕ_{21}=0.00	-0.1082	-0.1095	-0.0261	-0.1010	-0.0112	-0.1167	-0.1103	-0.0018	-0.1194
ϕ_{31}=0.00	-0.0118	0.0184	0.0092	-0.0067	-0.0088	0.0103	0.0121	-0.0133	0.0194
ϕ_{32}=0.00	-0.0009	-0.0049	0.0015	-0.0053	0.0017	-0.0031	-0.0037	-0.0069	0.0002
				$n = 200$					
No of cases converaged	98	94	99	100	99	100	99	99	99
True values of parameters									
b_{11}=0.65	0.6647	0.6881	0.6873	0.8059	0.6656	0.8291	0.8034	0.6469	0.8666
b_{21}=0.70	0.7157	0.7304	0.7319	0.8008	0.7156	0.8221	0.8052	0.7111	0.8429
b_{31}=0.75	0.7499	0.7698	0.7710	0.8539	0.7523	0.8781	0.8610	0.7433	0.9056
b_{41}=0.80	0.8029	0.8267	0.8269	0.9206	0.8050	0.9469	0.9277	0.7902	0.9771
b_{51}=0.85	0.8684	0.8266	0.8593	0.8581	0.8660	0.8760	0.9022	0.9852	0.9569
b_{61}=0.90	0.9011	0.8993	0.9027	0.9038	0.9022	0.9029	0.9034	0.8881	0.9134
b_{71}=0.95	0.9622	0.9531	0.9605	0.9612	0.9615	0.9635	0.9676	0.9525	0.9817
b_{81}=1.00	**1.0000**	0.9938	0.9983	0.9994	0.9999	1.0017	1.0117	1.0003	1.0319
b_{12}=1.00	**1.0000**	1.0353	1.0224	1.0187	1.0047	1.0361	1.0403	1.0086	1.0598
b_{22}=0.60	0.5971	0.6136	0.6055	0.6050	0.5998	0.6110	0.6120	0.5825	0.6226
b_{32}=0.70	0.7033	0.7224	0.7126	0.7121	0.7053	0.7197	0.7212	0.6927	0.7336
b_{42}=0.80	0.8053	0.8270	0.8160	0.8163	0.8077	0.8246	0.8269	0.7976	0.8414
b_{53}=1.00	**1.0000**	1.0402	1.0087	1.0059	1.0007	1.0413	1.0666	1.0017	1.0833
b_{63}=0.20	0.1936	0.2131	0.1871	0.1830	0.1919	0.1848	0.2132	0.1852	0.2142
b_{73}=0.30	0.2935	0.3134	0.2887	0.2846	0.2924	0.2885	0.3129	0.2852	0.3160
b_{83}=0.40	0.3996	0.4233	0.3962	0.3913	0.3984	0.3994	0.4257	0.3930	0.4307
r_f=0.50	**0.5000**	0.4964	0.4996	0.4982	0.4996	0.4994	0.4847	0.4985	0.4872
λ_1=1.00	0.9327	0.9347	0.9633	0.9760	0.9384	1.0050	0.9758	0.9935	0.9807
λ_2=2.00	2.0182	1.9278	1.9486	1.8423	2.0061	1.8451	1.8436	1.9837	1.7588
λ_3=3.00	3.0302	2.9305	2.9897	2.9936	3.0254	2.8477	2.7533	2.7368	2.6553
ϕ_{21}=0.00	0.0167	-0.0370	-0.0156	-0.1319	0.0147	-0.1683	-0.1578	0.0421	-0.2124
ϕ_{31}=0.00	0.0027	0.0018	0.0122	0.0174	0.0058	-0.0182	-0.0887	-0.1011	-0.1058
ϕ_{32}=0.00	-0.0501	-0.0232	-0.0411	-0.0463	-0.0491	-0.0330	-0.0237	-0.0696	-0.0385

Remarks: The values in boldface are fixed

Table A.4: Simulation result - Mean

Design D	Factor loadings = B1			Factor correlation = $\Phi 2$			$\lambda_1 = 1$ $\lambda_2 = 2$ $\lambda_3 = 0$		
Constraints	Exact	1a	1b	2a	2b	2c	3a	3b	3aR
				$n = 2000$					
No of cases converaged	100	100	99	100	100	99	100	100	100
True value of parameters									
b_{11}=0.65	0.7584	0.5565	0.6386	0.5800	0.6405	0.5836	0.5641	0.5885	0.5759
b_{21}=0.70	0.7722	0.6530	0.6989	0.6660	0.6988	0.6653	0.6562	0.6720	0.6638
b_{31}=0.75	0.8272	0.6894	0.7457	0.7049	0.7457	0.7068	0.6938	0.7108	0.7023
b_{41}=0.80	0.8907	0.7326	0.7946	0.7508	0.7952	0.7506	0.7378	0.7576	0.7465
b_{51}=0.85	0.8663	0.8566	0.8455	0.8531	0.8495	0.8402	0.8563	0.8627	0.8411
b_{61}=0.90	0.8914	0.8881	0.9011	0.8890	0.8987	0.8973	0.8878	0.8915	0.8799
b_{71}=0.95	0.9505	0.9470	0.9498	0.9491	0.9495	0.9454	0.9474	0.9505	0.9355
b_{81}=1.00	**1.0000**	0.9964	0.9993	0.9979	0.9998	0.9948	0.9970	0.9998	0.9803
b_{12}=1.00	**1.0000**	1.0041	1.0090	1.0089	1.0058	1.0030	1.0026	1.0072	1.0081
b_{22}=0.60	0.5932	0.5936	0.6004	0.5972	0.5995	0.5967	0.5933	0.5965	0.5965
b_{32}=0.70	0.6953	0.6965	0.7018	0.7008	0.7008	0.6976	0.6962	0.6992	0.7000
b_{42}=0.80	0.7929	0.7949	0.8030	0.7997	0.8015	0.7980	0.7945	0.7973	0.7991
b_{53}=1.00	**1.0000**	0.9947	0.9975	0.9960	0.9993	0.9998	1.0031	0.9979	1.0034
b_{63}=0.20	0.2244	0.2224	0.2103	0.2212	0.2090	0.2102	0.2185	0.2227	0.2189
b_{73}=0.30	0.3230	0.3234	0.3192	0.3205	0.3145	0.3187	0.3183	0.3218	0.3191
b_{83}=0.40	0.4259	0.4282	0.4225	0.4241	0.4156	0.4217	0.4224	0.4249	0.4238
r_f=0.50	**0.5000**	0.5144	0.4969	0.5198	0.4989	0.4971	0.5148	0.5015	0.5147
λ_1=1.00	1.0203	1.0105	1.0132	1.0007	1.0092	1.0145	1.0091	1.0194	1.0092
λ_2=2.00	1.8707	2.0618	1.9872	2.0297	1.9920	2.0542	2.0592	2.0319	2.0257
λ_3=0.00	-0.0356	-0.0355	-0.0069	-0.0275	-0.0083	-0.0029	-0.0357	-0.0328	-0.0009
ϕ_{21}=0.00	-0.1060	0.0964	0.0086	0.0688	0.0077	0.0659	0.0890	0.0603	0.0747
ϕ_{31}=0.00	-0.0243	-0.0108	-0.0013	-0.0073	-0.0025	0.0080	-0.0088	-0.0185	0.0198
ϕ_{32}=0.00	0.0036	-0.0166	-0.0007	-0.0101	-0.0028	-0.0022	-0.0216	-0.0088	-0.0218
				$n = 200$					
No of cases converaged	97	94	98	98	97	96	95	93	95
True values of parameters									
b_{11}=0.65	0.7780	0.5474	0.6298	0.8846	0.6352	0.8180	0.8143	0.8776	0.8611
b_{21}=0.70	0.7727	0.6621	0.7275	0.8504	0.7247	0.8223	0.8266	0.8288	0.8438
b_{31}=0.75	0.8085	0.6678	0.7347	0.9178	0.7515	0.8835	0.8947	0.8997	0.9050
b_{41}=0.80	0.9069	0.7519	0.8124	0.9870	0.8126	0.9481	0.9581	0.9718	0.9827
b_{51}=0.85	0.7242	0.7916	0.8732	0.8748	0.8812	0.8501	0.8430	0.8678	0.8539
b_{61}=0.90	0.9719	0.9485	0.9263	0.9046	0.9116	0.9198	0.9090	0.9105	0.9163
b_{71}=0.95	0.9776	0.9837	0.9784	0.9621	0.9631	0.9721	0.9651	0.9581	0.9573
b_{81}=1.00	**1.0000**	1.0053	0.9997	0.9999	1.0000	0.9944	0.9905	1.0001	0.9925
b_{12}=1.00	**1.0000**	1.0234	1.0231	1.0174	1.0045	1.0396	1.0460	1.0082	1.0543
b_{22}=0.60	0.5630	0.5522	0.5601	0.5723	0.5494	0.5843	0.5781	0.5518	0.5838
b_{32}=0.70	0.6787	0.6801	0.6959	0.7019	0.6643	0.7011	0.6958	0.6657	0.6963
b_{42}=0.80	0.7506	0.7391	0.7702	0.7750	0.7518	0.7925	0.7924	0.7560	0.7921
b_{53}=1.00	**1.0000**	1.0246	1.0089	1.0061	1.0008	1.0352	1.0514	1.0018	1.0541
b_{63}=0.20	0.1586	0.2846	0.1953	0.2080	0.2055	0.2245	0.2390	0.2065	0.2290
b_{73}=0.30	0.3259	0.4458	0.3168	0.3258	0.3279	0.3466	0.3501	0.3253	0.3497
b_{83}=0.40	0.4266	0.5458	0.4456	0.4591	0.4667	0.4714	0.4858	0.4615	0.4742
r_f=0.50	**0.5000**	0.4965	0.4973	0.4920	0.4977	0.4974	0.4744	0.4995	0.4821
λ_1=1.00	1.0661	1.1448	1.0875	1.0914	1.0901	1.0635	1.0821	1.0900	1.0999
λ_2=2.00	1.8289	2.0028	1.9334	1.6386	1.9492	1.6620	1.6701	1.6439	1.6625
λ_3=0.00	0.0078	-0.1675	-0.1298	-0.1302	-0.1413	-0.0962	-0.0851	-0.1315	-0.0896
ϕ_{21}=0.00	-0.0549	0.1821	0.0517	-0.1954	0.0593	-0.1702	-0.1757	-0.1814	-0.2033
ϕ_{31}=0.00	0.1565	0.0091	-0.0119	-0.0275	-0.0333	-0.0158	-0.0190	-0.0211	0.0119
ϕ_{32}=0.00	-0.0411	-0.0079	-0.0684	-0.0432	-0.0610	-0.0054	0.0035	-0.0484	-0.0542

Remarks: The values in boldface are fixed

Table A.5: Simulation result - Mean

Design E	Factor loadings = B2			Factor correlation = $\Phi 1$			$\lambda_1 = 1$ $\lambda_2 = 2$ $\lambda_3 = 3$		
Constraints	Exact	1a	1b	2a	2b	2c	3a	3b	3aR
				$n = 2000$					
No of cases converaged	96	87	100	94	100	89	89	93	94
True value of parameters									
b_{11}=0.30	0.2969	0.2976	0.2963	0.2974	0.2981	0.2896	0.2955	0.2981	0.2969
b_{21}=0.35	0.3492	0.3472	0.3479	0.3481	0.3496	0.3428	0.3451	0.3503	0.3484
b_{31}=0.40	0.4010	0.3996	0.3999	0.4013	0.4019	0.3942	0.3985	0.4020	0.4006
b_{41}=0.45	0.4462	0.4433	0.4448	0.4453	0.4473	0.4374	0.4424	0.4474	0.4457
b_{51}=0.50	0.4993	0.4861	0.4891	0.4920	0.4959	0.4829	0.4868	0.4939	0.4940
b_{61}=0.55	0.5451	0.5379	0.5404	0.5419	0.5439	0.5363	0.5377	0.5434	0.5399
b_{71}=0.60	0.6014	0.5952	0.5964	0.5982	0.6003	0.5916	0.5951	0.6016	0.5971
b_{81}=0.65	**0.6500**	0.6408	0.6436	0.6451	0.6488	0.6395	0.6413	0.6475	0.6445
b_{12}=1.00	**1.0000**	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.9976
b_{22}=0.60	0.5998	0.6007	0.6000	0.6009	0.5997	0.6016	0.6016	0.5993	0.5989
b_{32}=0.70	0.6971	0.6970	0.6969	0.6963	0.6964	0.6989	0.6973	0.6962	0.6946
b_{42}=0.80	0.8006	0.8011	0.8003	0.8005	0.7997	0.8024	0.8006	0.7991	0.7973
b_{53}=1.00	**1.0000**	1.0005	0.9995	0.9995	0.9995	1.0001	1.0025	0.9992	0.9983
b_{63}=0.20	0.2017	0.2018	0.2010	0.2013	0.2004	0.2034	0.2020	0.1997	0.2013
b_{73}=0.30	0.2994	0.2984	0.2985	0.2984	0.2979	0.3005	0.2990	0.2965	0.2977
b_{83}=0.40	0.4004	0.3995	0.3994	0.3994	0.3985	0.4009	0.4000	0.3979	0.3980
r_f=0.50	**0.5000**	0.4975	0.4984	0.4925	0.4975	0.4982	0.4919	0.4996	0.4874
λ_1=1.00	0.9904	1.0004	0.9994	1.0081	1.0034	0.9907	1.0069	1.0032	1.0157
λ_2=2.00	2.0023	2.0018	2.0021	2.0042	2.0001	2.0105	2.0054	1.9972	2.0166
λ_3=3.00	3.0085	3.0170	3.0183	3.0170	3.0092	3.0291	3.0126	3.0096	3.0213
ϕ_{21}=0.50	0.5005	0.5090	0.5044	0.5019	0.4992	0.5143	0.5084	0.4981	0.5085
ϕ_{31}=0.50	0.5040	0.5359	0.5270	0.5207	0.5106	0.5388	0.5343	0.5180	0.5299
ϕ_{32}=0.00	-0.0052	0.0002	-0.0017	-0.0033	-0.0053	0.0037	0.0009	-0.0050	-0.0123
				$n = 200$					
No of cases converaged	94	97	95	94	95	94	99	91	98
True values of parameters									
b_{11}=0.30	0.3222	0.3306	0.3286	0.3281	0.3315	0.3320	0.3298	0.3289	0.2678
b_{21}=0.35	0.3671	0.3743	0.3716	0.3649	0.3656	0.3733	0.3729	0.3671	0.3402
b_{31}=0.40	0.4098	0.4182	0.4152	0.4135	0.4117	0.4178	0.4144	0.4163	0.3787
b_{41}=0.45	0.4545	0.4642	0.4626	0.4562	0.4616	0.4639	0.4561	0.4626	0.4124
b_{51}=0.50	0.5072	0.4511	0.4921	0.4655	0.4862	0.4559	0.3903	0.4958	0.4843
b_{61}=0.55	0.5561	0.5451	0.5543	0.5531	0.5552	0.5509	0.5314	0.5548	0.5394
b_{71}=0.60	0.6114	0.5979	0.6094	0.6107	0.6042	0.6020	0.5821	0.6109	0.5965
b_{81}=0.65	**0.6500**	0.6379	0.6500	0.6510	0.6502	0.6416	0.6287	0.6503	0.6434
b_{12}=1.00	**1.0000**	1.0121	1.0046	1.0032	1.0008	1.0123	1.0163	1.0016	1.0215
b_{22}=0.60	0.6008	0.6015	0.6002	0.6024	0.6029	0.6022	0.6020	0.6014	0.6037
b_{32}=0.70	0.7027	0.7040	0.7041	0.7003	0.7054	0.7065	0.7040	0.7032	0.7042
b_{42}=0.80	0.8027	0.8030	0.8011	0.8010	0.8011	0.8048	0.8065	0.7999	0.8092
b_{53}=1.00	**1.0000**	1.0407	1.0089	1.0065	1.0009	1.0413	1.0611	1.0017	1.0498
b_{63}=0.20	0.2066	0.2153	0.2034	0.2006	0.2009	0.2147	0.2205	0.2022	0.2182
b_{73}=0.30	0.3090	0.3184	0.3076	0.3032	0.3077	0.3193	0.3249	0.3046	0.3177
b_{83}=0.40	0.4112	0.4226	0.4090	0.4020	0.4054	0.4256	0.4235	0.4081	0.4179
r_f=0.50	**0.5000**	0.4794	0.4974	0.4890	0.4973	0.4964	0.4435	0.4993	0.4525
λ_1=1.00	0.8090	0.9144	0.8465	0.8928	0.8335	0.7915	1.0266	0.8606	0.9895
λ_2=2.00	1.9954	1.9394	1.9761	1.9791	1.9801	1.9450	1.9423	1.9618	2.0203
λ_3=3.00	3.1414	3.0296	3.1150	3.1512	3.1561	3.0954	2.9997	3.1116	2.9846
ϕ_{21}=0.50	0.4822	0.4699	0.4698	0.4797	0.4743	0.4730	0.4958	0.4732	0.5907
ϕ_{31}=0.50	0.4783	0.5442	0.5034	0.5404	0.5132	0.5305	0.6223	0.4994	0.5992
ϕ_{32}=0.00	-0.0449	0.0005	-0.0340	-0.0357	-0.0402	-0.0088	0.0313	-0.0510	-0.0590

Remarks: The values in boldface are fixed

Table A.6: Simulation result - Mean

Design F	Factor loadings = B2			Factor correlation = $\Phi 1$			$\lambda_1 = 1$ $\lambda_2 = 2$ $\lambda_3 = 0$		
Constraints	Exact	1a	1b	2a	2b	2c	3a	3b	3aR
				$n = 2000$					
No of cases converaged	100	100	100	100	100	100	100	100	100
True value of parameters									
b_{11}=0.30	0.2987	0.3191	0.2962	0.2966	0.2884	0.3108	0.3693	0.3036	0.2702
b_{21}=0.35	0.3692	0.4038	0.3775	0.3861	0.3637	0.4019	0.4527	0.3786	0.3737
b_{31}=0.40	0.4156	0.4575	0.4389	0.4365	0.4211	0.4655	0.5113	0.4296	0.4211
b_{41}=0.45	0.4667	0.5122	0.4833	0.4881	0.4644	0.5141	0.5720	0.4814	0.4695
b_{51}=0.50	0.4750	0.5284	0.5072	0.5049	0.4969	0.5369	0.5468	0.4799	0.5521
b_{61}=0.55	0.5557	0.5727	0.5635	0.5516	0.5534	0.5790	0.5825	0.5576	0.5893
b_{71}=0.60	0.6085	0.6355	0.6152	0.6111	0.6045	0.6340	0.6481	0.6114	0.6558
b_{81}=0.65	**0.6500**	0.6884	0.6647	0.6611	0.6526	0.6866	0.7036	0.6542	0.7122
b_{12}=1.00	**1.0000**	0.9998	1.0000	1.0000	1.0000	1.0001	1.0001	1.0000	1.0383
b_{22}=0.60	0.5826	0.5537	0.5701	0.5601	0.5777	0.5507	0.5122	0.5743	0.5582
b_{32}=0.70	0.6822	0.6477	0.6580	0.6556	0.6686	0.6372	0.6027	0.6687	0.6534
b_{42}=0.80	0.7803	0.7461	0.7619	0.7552	0.7729	0.7384	0.6971	0.7671	0.7542
b_{53}=1.00	**1.0000**	1.0305	1.0176	1.0137	1.0035	1.0330	1.0490	1.0069	1.0559
b_{63}=0.20	0.1836	0.1837	0.1993	0.2049	0.2053	0.1888	0.1809	0.1860	0.1908
b_{73}=0.30	0.2922	0.2882	0.3056	0.3122	0.3112	0.2932	0.2842	0.2947	0.2968
b_{83}=0.40	0.3997	0.3873	0.4071	0.4146	0.4135	0.3923	0.3815	0.4018	0.3932
r_f=0.50	**0.5000**	0.5364	0.5011	0.5308	0.5007	0.5021	0.5363	0.5029	0.5406
λ_1=1.00	1.0474	1.0108	1.0555	1.0370	1.0643	1.0579	1.0259	1.0506	1.0067
λ_2=2.00	2.0036	1.9585	1.9975	1.9783	2.0022	1.9815	1.8694	1.9933	1.8844
λ_3=0.00	-0.0240	-0.0968	-0.0577	-0.0865	-0.0546	-0.0861	-0.1155	-0.0349	-0.1260
ϕ_{21}=0.50	0.5089	0.4929	0.5191	0.5282	0.5361	0.5031	0.4446	0.5047	0.5219
ϕ_{31}=0.50	0.5389	0.4469	0.4718	0.4717	0.4917	0.4336	0.4141	0.5244	0.4102
ϕ_{32}=0.00	-0.0247	-0.0540	-0.0312	-0.0401	-0.0308	-0.0458	-0.0648	-0.0260	-0.0194
				$n = 200$					
No of cases converaged	95	95	94	94	95	96	95	96	96
True values of parameters									
b_{11}=0.30	0.3390	0.3366	0.3373	0.3360	0.3398	0.3482	0.3408	0.3393	0.2511
b_{21}=0.35	0.4207	0.4710	0.4090	0.3963	0.4219	0.4881	0.4836	0.4230	0.4236
b_{31}=0.40	0.4859	0.5203	0.4856	0.4689	0.4857	0.5214	0.5049	0.4877	0.4560
b_{41}=0.45	0.5154	0.5459	0.5265	0.5171	0.5176	0.5645	0.5758	0.5185	0.4935
b_{51}=0.50	0.4902	0.5359	0.5084	0.4727	0.4902	0.5511	0.5745	0.4943	0.6062
b_{61}=0.55	0.5525	0.5491	0.5552	0.5472	0.5524	0.5601	0.5383	0.5530	0.5547
b_{71}=0.60	0.6099	0.6228	0.6121	0.6154	0.6102	0.6277	0.6123	0.6099	0.6333
b_{81}=0.65	**0.6500**	0.6641	0.6540	0.6530	0.6505	0.6638	0.6608	0.6509	0.6836
b_{12}=1.00	**1.0000**	1.0108	1.0042	1.0027	1.0005	1.0140	1.0177	1.0010	1.0465
b_{22}=0.60	0.5255	0.4872	0.5450	0.5573	0.5252	0.4834	0.4796	0.5237	0.5019
b_{32}=0.70	0.6075	0.5848	0.6168	0.6303	0.6082	0.5943	0.6046	0.6058	0.6099
b_{42}=0.80	0.7211	0.7106	0.7226	0.7329	0.7202	0.7010	0.6803	0.7179	0.7131
b_{53}=1.00	**1.0000**	1.0455	1.0166	1.0102	1.0024	1.0542	1.0655	1.0045	1.0750
b_{63}=0.20	0.2416	0.2534	0.2474	0.2402	0.2420	0.2371	0.2494	0.2447	0.2506
b_{73}=0.30	0.3410	0.3477	0.3486	0.3325	0.3415	0.3331	0.3393	0.3430	0.3413
b_{83}=0.40	0.4440	0.4462	0.4495	0.4349	0.4444	0.4433	0.4343	0.4469	0.4366
r_f=0.50	**0.5000**	0.5205	0.4997	0.5212	0.5002	0.4993	0.5099	0.5000	0.5155
λ_1=1.00	1.1873	1.1543	1.1866	1.1173	1.1847	1.1868	1.1833	1.1981	1.1618
λ_2=2.00	1.8906	1.8795	1.8948	1.8846	1.8870	1.8839	1.8677	1.8854	1.8834
λ_3=0.00	-0.1453	-0.1813	-0.1601	-0.1097	-0.1427	-0.1823	-0.2124	-0.1584	-0.2376
ϕ_{21}=0.50	0.5340	0.5383	0.5324	0.5344	0.5319	0.5005	0.5528	0.5336	0.6479
ϕ_{31}=0.50	0.4581	0.3893	0.4186	0.4855	0.4569	0.3545	0.3910	0.4474	0.3792
ϕ_{32}=0.00	-0.0709	-0.0845	-0.0786	-0.0670	-0.0697	-0.0854	-0.0691	-0.0732	-0.0893

Remarks: The values in boldface are fixed

Table A.7: Simulation result - Mean

Design G	Factor loadings = B2			Factor correlation = $\Phi 2$			$\lambda_1 = 1$ $\lambda_2 = 2$ $\lambda_3 = 3$		
Constraints	Exact	1a	1b	2a	2b	2c	3a	3b	3aR
$n = 2000$									
No of cases converaged	100	99	99	100	99	100	96	100	96
True value of parameters									
b_{11}=0.30	0.4203	0.2331	0.3045	0.2486	0.2994	0.2666	0.2585	0.4053	0.2570
b_{21}=0.35	0.4294	0.3171	0.3561	0.3247	0.3524	0.3320	0.3326	0.4212	0.3312
b_{31}=0.40	0.4869	0.3578	0.4042	0.3671	0.3997	0.3766	0.3750	0.4776	0.3740
b_{41}=0.45	0.5518	0.4032	0.4549	0.4140	0.4501	0.4233	0.4244	0.5410	0.4226
b_{51}=0.50	0.5092	0.4933	0.4881	0.5029	0.5032	0.4924	0.4982	0.5082	0.4782
b_{61}=0.55	0.5345	0.5316	0.5449	0.5343	0.5478	0.5445	0.5312	0.5347	0.5267
b_{71}=0.60	0.5958	0.5890	0.5948	0.5927	0.5992	0.5947	0.5887	0.5955	0.5821
b_{81}=0.65	**0.6500**	0.6453	0.6439	0.6489	0.6494	0.6435	0.6448	0.6504	0.6356
b_{12}=1.00	**1.0000**	1.0089	1.0083	1.0078	1.0054	1.0074	1.0062	1.0065	1.0113
b_{22}=0.60	0.6030	0.6053	0.6041	0.6054	0.6030	0.6037	0.6041	0.6050	0.6074
b_{32}=0.70	0.7048	0.7078	0.7059	0.7075	0.7046	0.7052	0.7077	0.7071	0.7113
b_{42}=0.80	0.8027	0.8076	0.8067	0.8069	0.8051	0.8060	0.8071	0.8054	0.8112
b_{53}=1.00	**1.0000**	0.9995	1.0019	0.9981	0.9998	1.0027	0.9913	0.9987	0.9938
b_{63}=0.20	0.2144	0.2111	0.2014	0.2096	0.2013	0.2018	0.2105	0.2114	0.2111
b_{73}=0.30	0.3130	0.3107	0.3014	0.3091	0.3008	0.3017	0.3098	0.3100	0.3106
b_{83}=0.40	0.4113	0.4067	0.4020	0.4060	0.4015	0.4025	0.4056	0.4077	0.4067
r_f=0.50	**0.5000**	0.5032	0.5015	0.5056	0.5015	0.5013	0.5101	0.5000	0.5104
λ_1=1.00	0.9334	0.9488	0.9896	0.9507	0.9882	0.9897	0.9331	0.9513	0.9326
λ_2=2.00	1.9053	2.0467	1.9878	2.0267	1.9981	2.0145	2.0367	1.9022	2.0188
λ_3=3.00	3.0230	3.0417	3.0209	3.0246	3.0060	3.0109	3.0565	3.0202	3.0669
ϕ_{21}=0.00	-0.1178	0.0636	-0.0062	0.0503	0.0002	0.0316	0.0381	-0.1057	0.0385
ϕ_{31}=0.00	-0.0156	0.0093	0.0191	-0.0017	0.0021	0.0152	0.0096	-0.0105	0.0371
ϕ_{32}=0.00	-0.0022	-0.0106	0.0031	-0.0078	0.0015	-0.0053	-0.0104	-0.0035	-0.0134
$n = 200$									
No of cases converaged	99	97	99	99	99	97	97	96	96
True values of parameters									
b_{11}=0.30	0.2952	0.3017	0.2866	0.4811	0.2924	0.4808	0.3119	0.2866	0.3101
b_{21}=0.35	0.3567	0.3687	0.3591	0.4750	0.3557	0.4840	0.3769	0.3530	0.3710
b_{31}=0.40	0.3851	0.3985	0.3832	0.5245	0.3838	0.5355	0.4059	0.3771	0.3989
b_{41}=0.45	0.4347	0.4486	0.4330	0.5954	0.4334	0.6035	0.4618	0.4264	0.4535
b_{51}=0.50	0.5186	0.4905	0.4996	0.4967	0.5173	0.5002	0.5046	0.5003	0.5096
b_{61}=0.55	0.5589	0.5499	0.5603	0.5583	0.5590	0.5519	0.5513	0.5586	0.5476
b_{71}=0.60	0.6178	0.6103	0.6177	0.6173	0.6178	0.6133	0.6149	0.6193	0.6090
b_{81}=0.65	**0.6500**	0.6431	0.6511	0.6510	0.6503	0.6429	0.6499	0.6507	0.6447
b_{12}=1.00	**1.0000**	1.0345	1.0216	1.0175	1.0045	1.0363	1.0391	1.0075	1.0447
b_{22}=0.60	0.6014	0.6098	0.6073	0.6059	0.6027	0.6131	0.6130	0.6021	0.6172
b_{32}=0.70	0.7061	0.7178	0.7140	0.7121	0.7079	0.7192	0.7206	0.7082	0.7262
b_{42}=0.80	0.8108	0.8229	0.8190	0.8163	0.8125	0.8251	0.8262	0.8136	0.8321
b_{53}=1.00	**1.0000**	1.0469	1.0085	1.0053	1.0007	1.0492	1.0648	1.0013	1.0689
b_{63}=0.20	0.1897	0.1861	0.1821	0.1835	0.1881	0.1874	0.1910	0.1830	0.1870
b_{73}=0.30	0.2904	0.2863	0.2860	0.2867	0.2887	0.2866	0.2867	0.2813	0.2849
b_{83}=0.40	0.3974	0.3954	0.3939	0.3916	0.3958	0.3990	0.3997	0.3891	0.3960
r_f=0.50	**0.5000**	0.4940	0.4988	0.4977	0.4995	0.4978	0.4939	0.4997	0.4974
λ_1=1.00	0.9430	1.0190	0.9984	1.0101	0.9550	0.9848	1.0092	0.9794	1.0280
λ_2=2.00	2.0209	1.9408	1.9813	1.8341	2.0124	1.8817	1.9640	2.0024	1.9425
λ_3=3.00	3.0339	2.8651	2.9732	2.9863	3.0271	2.8454	2.8222	3.0334	2.7970
ϕ_{21}=0.00	0.0450	0.0107	0.0371	-0.1625	0.0449	-0.1754	0.0029	0.0519	0.0388
ϕ_{31}=0.00	0.0283	0.0259	0.0367	0.0430	0.0304	0.0082	-0.0022	0.0563	0.0704
ϕ_{32}=0.00	-0.0498	-0.0171	-0.0353	-0.0452	-0.0483	-0.0295	-0.0172	-0.0526	-0.0161

Remarks: The values in boldface are fixed

Table A.8: Simulation result - Mean

Design H	Factor loadings = B2			Factor correlation = $\Phi 2$			$\lambda_1=1$ $\lambda_2=2$ $\lambda_3=0$		
Constraints	Exact	1a	1b	2a	2b	2c	3a	3b	3aR
				$n=2000$					
No of cases converaged	100	100	98	100	100	100	100	100	99
True value of parameters									
b_{11}=0.30	0.3793	0.4144	0.2964	0.4190	0.2963	0.4042	0.4208	0.3872	0.4787
b_{21}=0.35	0.4079	0.4369	0.3616	0.4409	0.3560	0.4305	0.4599	0.4152	0.4883
b_{31}=0.40	0.4647	0.4997	0.4115	0.5066	0.4045	0.4922	0.5283	0.4733	0.5616
b_{41}=0.45	0.5263	0.5663	0.4652	0.5752	0.4561	0.5564	0.5973	0.5365	0.6370
b_{51}=0.50	0.5087	0.5105	0.5016	0.5273	0.4909	0.5062	0.5386	0.5100	0.5659
b_{61}=0.55	0.5415	0.5366	0.5507	0.5366	0.5514	0.5482	0.5326	0.5407	0.5405
b_{71}=0.60	0.6008	0.5972	0.6005	0.5996	0.6002	0.5985	0.5950	0.6009	0.6042
b_{81}=0.65	**0.6500**	0.6479	0.6517	0.6529	0.6501	0.6519	0.6524	0.6505	0.6617
b_{12}=1.00	**1.0000**	1.0133	1.0107	1.0123	1.0062	1.0170	1.0314	1.0084	1.0377
b_{22}=0.60	0.5886	0.5763	0.5869	0.5673	0.5951	0.5795	0.5362	0.5901	0.5585
b_{32}=0.70	0.6859	0.6721	0.6857	0.6596	0.6950	0.6762	0.6232	0.6877	0.6503
b_{42}=0.80	0.7818	0.7660	0.7819	0.7508	0.7939	0.7717	0.7125	0.7842	0.7414
b_{53}=1.00	**1.0000**	1.0088	0.9976	0.9947	0.9996	1.0052	1.0141	0.9987	1.0244
b_{63}=0.20	0.2182	0.2379	0.2216	0.2501	0.2052	0.2290	0.2767	0.2230	0.2625
b_{73}=0.30	0.3175	0.3420	0.3321	0.3546	0.3125	0.3394	0.3849	0.3234	0.3701
b_{83}=0.40	0.4220	0.4509	0.4372	0.4634	0.4148	0.4440	0.4949	0.4288	0.4786
r_f=0.50	**0.5000**	0.5036	0.4945	0.5045	0.4972	0.4946	0.4952	0.5021	0.4968
λ_1=1.00	1.0550	1.1119	1.0727	1.1433	1.0327	1.1014	1.2374	1.0602	1.1988
λ_2=2.00	1.8915	1.7956	1.9618	1.7725	1.9822	1.8288	1.7305	1.8619	1.6810
λ_3=0.00	-0.0444	-0.0965	-0.0483	-0.1359	-0.0116	-0.0755	-0.2037	-0.0523	-0.1937
ϕ_{21}=0.00	-0.0688	-0.0898	0.0159	-0.0837	0.0069	-0.0852	-0.0599	-0.0802	-0.1207
ϕ_{31}=0.00	-0.0078	-0.0244	-0.0124	-0.0510	0.0126	-0.0226	-0.0913	-0.0136	-0.0958
ϕ_{32}=0.00	-0.0050	-0.0059	-0.0096	-0.0019	-0.0090	-0.0047	-0.0171	0.0017	-0.0074
				$n=200$					
No of cases converaged	96	93	92	94	93	90	91	96	91
True values of parameters									
b_{11}=0.30	0.3332	0.2955	0.3171	0.4723	0.3588	0.5134	0.2632	0.3333	0.2790
b_{21}=0.35	0.3944	0.3787	0.3920	0.4794	0.4121	0.5089	0.3662	0.4018	0.4095
b_{31}=0.40	0.4233	0.4079	0.4213	0.5185	0.4444	0.5737	0.3890	0.4230	0.4451
b_{41}=0.45	0.5203	0.4913	0.5030	0.5919	0.5097	0.6451	0.4729	0.5081	0.5153
b_{51}=0.50	0.4785	0.5999	0.5804	0.5569	0.5356	0.5531	0.5453	0.5228	0.5840
b_{61}=0.55	0.5667	0.5326	0.5380	0.5496	0.5979	0.5349	0.5484	0.5642	0.5939
b_{71}=0.60	0.6455	0.6239	0.6329	0.6146	0.6549	0.6194	0.6469	0.6547	0.6773
b_{81}=0.65	**0.6500**	0.6504	0.6492	0.6506	0.6501	0.6448	0.6586	0.6503	0.6687
b_{12}=1.00	**1.0000**	1.0408	1.0225	1.0183	1.0044	1.0408	1.0484	1.0076	1.0580
b_{22}=0.60	0.5397	0.5686	0.5540	0.5609	0.5422	0.5484	0.5839	0.5412	0.5480
b_{32}=0.70	0.6513	0.6817	0.6707	0.6798	0.6546	0.6620	0.7025	0.6616	0.6559
b_{42}=0.80	0.7126	0.7557	0.7454	0.7579	0.7449	0.7238	0.7743	0.7291	0.7469
b_{53}=1.00	**1.0000**	1.0287	1.0095	1.0055	1.0009	1.0248	1.0437	1.0016	1.0573
b_{63}=0.20	0.1975	0.1863	0.1963	0.1983	0.1408	0.2451	0.1954	0.2081	0.1663
b_{73}=0.30	0.2783	0.2593	0.2712	0.2863	0.2509	0.3281	0.2638	0.2890	0.2516
b_{83}=0.40	0.4360	0.4135	0.4235	0.4176	0.4298	0.4654	0.4134	0.4494	0.4320
r_f=0.50	**0.5000**	0.4927	0.4967	0.4961	0.4963	0.4978	0.4807	0.4993	0.4845
λ_1=1.00	1.1534	1.1437	1.1443	1.0995	1.1383	1.2035	1.1333	1.2113	1.1557
λ_2=2.00	1.8768	1.8865	1.8801	1.6936	1.8529	1.5735	1.9302	1.8698	1.8868
λ_3=0.00	-0.1180	-0.1996	-0.1909	-0.1299	-0.1541	-0.2169	-0.1496	-0.2044	-0.1953
ϕ_{21}=0.00	0.0365	0.0488	0.0246	-0.1345	0.0007	-0.1412	0.0504	0.0550	0.0473
ϕ_{31}=0.00	0.0535	-0.0434	-0.0234	-0.0038	-0.0024	-0.0539	-0.0132	-0.0051	-0.0442
ϕ_{32}=0.00	-0.0614	-0.0565	-0.0470	-0.0311	-0.0446	-0.0297	-0.0416	-0.0706	-0.0570

Remarks: The values in boldface are fixed

Appendix B

Simulation result - Bias

Table B.1: Simulation result - Bias

Design A	Factor loadings = B1		Factor correlation = $\Phi 1$			$\lambda_1 = 1$ $\lambda_2 = 2$ $\lambda_3 = 3$		
Constraints	Exact	1a	1b	2a	2b	2c	3a	3b	3aR
$n = 2000$									
No of cases converaged	98	97	99	94	100	92	95	95	99
True value of parameters									
b_{11}=0.65	-0.0016	-0.0065	-0.0056	-0.0048	-0.0031	-0.0075	-0.0071	-0.0048	-0.0032
b_{21}=0.70	0.0101	-0.0044	-0.0037	-0.0032	-0.0017	-0.0056	-0.0055	-0.0039	-0.0020
b_{31}=0.75	0.0028	-0.0026	-0.0019	-0.0004	0.0004	-0.0047	-0.0032	-0.0011	-0.0002
b_{41}=0.80	0.0045	-0.0059	-0.0057	-0.0043	-0.0031	-0.0080	-0.0066	-0.0050	-0.0032
b_{51}=0.85	-0.0074	-0.0120	-0.0095	-0.0105	-0.0026	-0.0124	-0.0156	-0.0026	-0.0059
b_{61}=0.90	-0.0120	-0.0104	-0.0089	-0.0090	-0.0059	-0.0109	-0.0110	-0.0066	-0.0092
b_{71}=0.95	-0.0049	-0.0052	-0.0041	-0.0042	-0.0005	-0.0067	-0.0070	-0.0006	-0.0045
b_{81}=1.00	**0.0000**	-0.0079	-0.0063	-0.0053	-0.0016	-0.0085	-0.0096	-0.0026	-0.0057
b_{12}=1.00	**0.0000**	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	-0.0028
b_{22}=0.60	-0.0089	0.0003	0.0000	-0.0003	-0.0004	0.0010	0.0009	-0.0001	-0.0015
b_{32}=0.70	-0.0054	-0.0029	-0.0029	-0.0045	-0.0034	-0.0020	-0.0024	-0.0032	-0.0048
b_{42}=0.80	-0.0071	-0.0006	-0.0002	-0.0019	-0.0007	0.0008	0.0000	-0.0007	-0.0027
b_{53}=1.00	**0.0000**	0.0001	-0.0010	-0.0010	-0.0007	-0.0002	-0.0010	-0.0010	-0.0026
b_{63}=0.20	-0.0009	0.0017	0.0012	0.0000	0.0006	0.0027	0.0017	0.0007	0.0016
b_{73}=0.30	-0.0015	-0.0004	-0.0009	-0.0023	-0.0014	0.0008	-0.0004	-0.0017	-0.0007
b_{83}=0.40	-0.0048	0.0002	-0.0002	-0.0019	-0.0009	0.0016	0.0001	-0.0007	-0.0005
r_f=0.50	**0.0000**	-0.0069	-0.0009	-0.0103	-0.0018	0.0008	-0.0038	-0.0003	-0.0056
λ_1=1.00	0.0169	0.0029	-0.0017	0.0134	0.0011	-0.0087	-0.0022	0.0002	-0.0004
λ_2=2.00	-0.0018	0.0096	0.0052	0.0054	0.0020	0.0083	0.0079	0.0033	0.0143
λ_3=3.00	-0.0084	0.0184	0.0163	0.0127	0.0050	0.0206	0.0273	0.0051	0.0234
ϕ_{21}=0.50	-0.0037	0.0027	0.0014	0.0002	-0.0024	0.0040	0.0033	-0.0004	0.0009
ϕ_{31}=0.50	0.0118	0.0212	0.0161	0.0164	0.0040	0.0190	0.0275	0.0051	0.0160
ϕ_{32}=0.00	-0.0081	-0.0009	-0.0020	-0.0040	-0.0048	0.0011	0.0007	-0.0045	-0.0118
$n = 200$									
No of cases converaged	97	96	98	98	98	99	97	98	98
True values of parameters									
b_{11}=0.65	0.0125	0.0273	0.0219	0.0305	0.0232	0.0601	0.0331	0.0157	0.0113
b_{21}=0.70	0.0084	0.0218	0.0186	0.0506	0.0155	0.0428	0.0255	0.0117	0.0172
b_{31}=0.75	0.0003	0.0128	0.0111	0.0575	0.0093	0.0395	0.0193	0.0033	0.0098
b_{41}=0.80	-0.0037	0.0077	0.0071	0.1349	0.0047	0.0393	0.0131	-0.0013	0.0032
b_{51}=0.85	-0.0269	-0.0534	-0.0329	-0.0013	-0.0237	-0.0243	-0.0639	-0.0253	0.0355
b_{61}=0.90	-0.0017	-0.0013	0.0009	0.0523	-0.0008	0.0014	-0.0122	-0.0005	-0.0011
b_{71}=0.95	0.0030	-0.0038	0.0041	-0.0099	0.0042	0.0037	-0.0084	0.0043	0.0117
b_{81}=1.00	**0.0000**	-0.0113	-0.0013	0.0187	0.0001	-0.0007	-0.0134	0.0002	0.0131
b_{12}=1.00	**0.0000**	0.0082	0.0045	0.0000	0.0008	0.0093	0.0122	0.0015	0.0277
b_{22}=0.60	0.0018	0.0004	-0.0001	-0.0128	0.0017	0.0016	-0.0001	0.0016	0.0089
b_{32}=0.70	0.0052	0.0047	0.0032	0.0364	0.0049	0.0052	0.0031	0.0054	0.0138
b_{42}=0.80	0.0007	0.0034	0.0001	-0.0574	0.0017	0.0034	0.0019	0.0019	0.0134
b_{53}=1.00	**0.0000**	0.0293	0.0062	0.0147	0.0006	0.0329	0.0646	0.0012	0.0568
b_{63}=0.20	0.0092	0.0006	0.0030	-0.0168	0.0083	0.0045	0.0151	0.0083	0.0104
b_{73}=0.30	0.0108	0.0073	0.0054	-0.0297	0.0095	0.0105	0.0188	0.0101	0.0163
b_{83}=0.40	0.0056	0.0062	0.0030	-0.0083	0.0056	0.0101	0.0175	0.0056	0.0147
r_f=0.50	**0.0000**	-0.0064	-0.0007	-0.0037	-0.0007	-0.0014	-0.0349	-0.0001	-0.0308
λ_1=1.00	-0.1188	-0.0417	-0.0721	-0.1330	-0.1127	-0.0557	-0.0134	-0.1139	-0.0177
λ_2=2.00	0.0205	-0.0423	-0.0134	-0.2583	0.0026	-0.0509	-0.0462	0.0123	-0.0267
λ_3=3.00	0.1778	0.0693	0.1157	-0.2950	0.1695	0.0352	0.0200	0.1673	-0.0502
ϕ_{21}=0.50	-0.0100	-0.0272	-0.0278	-0.0666	-0.0230	-0.0666	-0.0401	-0.0155	-0.0230
ϕ_{31}=0.50	0.0214	0.0650	0.0297	-0.0102	0.0185	0.0198	0.0605	0.0182	-0.0102
ϕ_{32}=0.00	-0.0294	0.0040	-0.0152	0.1434	-0.0457	-0.0142	0.0301	-0.0277	-0.0500

Remarks: The values in boldface are fixed

Table B.2: Simulation result - Bias

Design B	Factor loadings = B1			Factor correlation = $\Phi 1$			$\lambda_1 = 1$ $\lambda_2 = 2$ $\lambda_3 = 0$		
Constraints	Exact	1a	1b	2a	2b	2c	3a	3b	3aR
					$n = 2000$				
No of cases converaged	100	100	100	100	100	100	100	100	100
True value of parameters									
b_{11}=0.65	0.0005	0.0251	-0.0008	0.0066	-0.0123	0.0046	0.0305	0.0044	-0.0427
b_{21}=0.70	0.0177	0.0665	0.0137	0.0372	0.0075	0.0291	0.0714	0.0195	0.0310
b_{31}=0.75	0.0115	0.0638	0.0162	0.0340	0.0082	0.0313	0.0689	0.0138	0.0206
b_{41}=0.80	0.0121	0.0673	0.0113	0.0350	0.0024	0.0279	0.0728	0.0148	0.0173
b_{51}=0.85	-0.0276	0.0145	0.0023	-0.0077	-0.0049	0.0249	0.0277	-0.0248	0.0270
b_{61}=0.90	0.0076	0.0248	0.0142	0.0101	0.0049	0.0216	0.0301	0.0110	0.0348
b_{71}=0.95	0.0085	0.0330	0.0148	0.0166	0.0049	0.0242	0.0408	0.0122	0.0456
b_{81}=1.00	**0.0000**	0.0317	0.0130	0.0129	0.0027	0.0251	0.0412	0.0042	0.0459
b_{12}=1.00	**0.0000**	0.0001	0.0000	0.0000	0.0000	0.0002	0.0006	0.0000	0.0312
b_{22}=0.60	-0.0145	-0.0551	-0.0141	-0.0352	-0.0162	-0.0257	-0.0570	-0.0137	-0.0420
b_{32}=0.70	-0.0116	-0.0540	-0.0167	-0.0355	-0.0182	-0.0275	-0.0551	-0.0108	-0.0367
b_{42}=0.80	-0.0125	-0.0530	-0.0123	-0.0339	-0.0140	-0.0237	-0.0535	-0.0116	-0.0321
b_{53}=1.00	**0.0000**	0.0315	0.0116	0.0144	0.0023	0.0211	0.0385	0.0033	0.0469
b_{63}=0.20	-0.0178	-0.0171	-0.0079	-0.0084	0.0024	-0.0104	-0.0227	-0.0213	-0.0235
b_{73}=0.30	-0.0083	-0.0104	-0.0034	-0.0014	0.0072	-0.0070	-0.0179	-0.0120	-0.0182
b_{83}=0.40	-0.0028	-0.0102	-0.0032	0.0008	0.0082	-0.0088	-0.0194	-0.0070	-0.0190
r_f=0.50	**0.0000**	0.0308	0.0013	0.0296	-0.0006	0.0021	0.0450	0.0019	0.0459
λ_1=1.00	0.0242	0.0129	0.0201	0.0133	0.0339	0.0245	-0.0054	0.0181	-0.0065
λ_2=2.00	-0.0003	-0.0442	-0.0049	-0.0277	-0.0005	-0.0097	-0.0510	-0.0021	-0.0405
λ_3=0.00	-0.0092	-0.0730	-0.0341	-0.0552	-0.0396	-0.0582	-0.0794	-0.0083	-0.0867
ϕ_{21}=0.50	-0.0052	-0.0266	-0.0006	-0.0055	0.0177	-0.0044	-0.0341	-0.0113	0.0222
ϕ_{31}=0.50	0.0324	-0.0347	-0.0116	-0.0047	-0.0039	-0.0389	-0.0485	0.0288	-0.0521
ϕ_{32}=0.00	-0.0231	-0.0497	-0.0268	-0.0355	-0.0248	-0.0378	-0.0545	-0.0244	-0.0235
					$n = 200$				
No of cases converaged	100	100	98	99	98	98	95	100	100
True values of parameters									
b_{11}=0.65	0.0177	0.0338	0.0411	0.0996	0.0135	0.0891	0.0076	0.0550	-0.0101
b_{21}=0.70	0.0638	0.0644	0.0882	0.1100	0.0691	0.1319	0.0934	0.0925	0.0504
b_{31}=0.75	0.0507	0.0268	0.0766	0.1043	0.0578	0.1223	0.0712	0.0804	0.0296
b_{41}=0.80	0.0309	0.0238	0.0608	0.1004	0.0424	0.1091	0.0685	0.0681	0.0142
b_{51}=0.85	-0.0553	-0.1012	-0.0126	-0.0312	-0.0490	0.0438	0.0633	-0.0608	0.0889
b_{61}=0.90	0.0045	0.0247	0.0085	0.0112	0.0031	0.0316	-0.0122	0.0074	0.0269
b_{71}=0.95	0.0104	0.0332	0.0148	0.0220	0.0084	0.0442	0.0128	0.0114	0.0460
b_{81}=1.00	**0.0000**	0.0095	0.0068	0.0041	0.0009	0.0272	0.0089	0.0016	0.0497
b_{12}=1.00	**0.0000**	0.0038	0.0036	0.0029	0.0005	0.0152	0.0334	0.0010	0.0529
b_{22}=0.60	-0.0695	-0.0587	-0.0845	-0.0794	-0.0762	-0.0882	-0.1001	-0.0889	-0.0424
b_{32}=0.70	-0.0630	-0.0244	-0.0778	-0.0735	-0.0707	-0.0784	-0.0810	-0.0820	-0.0245
b_{42}=0.80	-0.0473	-0.0145	-0.0650	-0.0688	-0.0597	-0.0634	-0.0790	-0.0723	-0.0123
b_{53}=1.00	**0.0000**	0.0399	0.0176	0.0122	0.0021	0.0534	0.0660	0.0040	0.0861
b_{63}=0.20	0.0325	0.0082	0.0400	0.0171	0.0355	0.0215	0.0913	0.0295	0.0067
b_{73}=0.30	0.0342	-0.0025	0.0444	0.0132	0.0381	0.0151	0.0814	0.0301	-0.0018
b_{83}=0.40	0.0294	-0.0211	0.0398	0.0203	0.0321	0.0224	0.0806	0.0285	-0.0092
r_f=0.50	**0.0000**	0.0538	-0.0006	0.0008	0.0005	-0.0007	-0.0039	0.0000	0.0376
λ_1=1.00	0.1124	-0.0228	0.1227	0.0969	0.1184	0.1056	0.0864	0.1115	0.0344
λ_2=2.00	-0.0935	-0.0801	-0.1340	-0.1999	-0.0908	-0.1729	-0.1833	-0.1500	-0.1538
λ_3=0.00	-0.0927	-0.0127	-0.1361	-0.0963	-0.1066	-0.1615	-0.1336	-0.0843	-0.1547
ϕ_{21}=0.50	0.0056	-0.0238	-0.0172	-0.0883	0.0125	-0.0803	-0.0275	-0.0353	0.0111
ϕ_{31}=0.50	0.0219	0.1326	-0.0391	0.0032	0.0114	-0.1006	-0.1148	0.0280	-0.0949
ϕ_{32}=0.00	-0.0439	0.0034	-0.0453	-0.0409	-0.0471	-0.0653	0.0115	-0.0405	-0.0575

Remarks: The values in boldface are fixed

Table B.3: Simulation result - Bias

Design C	Factor loadings = B1			Factor correlation = $\Phi 2$			$\lambda_1 = 1$ $\lambda_2 = 2$ $\lambda_3 = 3$		
Constraints	Exact	1a	1b	2a	2b	2c	3a	3b	3aR
				$n = 2000$					
No of cases converaged	100	100	100	92	99	100	100	100	100
True value of parameters									
b_{11}=0.65	0.1087	0.1063	0.0228	0.0962	0.0095	0.1138	0.1073	-0.0006	0.1177
b_{21}=0.70	0.0671	0.0654	0.0165	0.0645	0.0080	0.0702	0.0664	0.0059	0.0721
b_{31}=0.75	0.0757	0.0748	0.0168	0.0724	0.0069	0.0800	0.0755	0.0010	0.0826
b_{41}=0.80	0.0864	0.0859	0.0194	0.0800	0.0079	0.0915	0.0865	0.0035	0.0951
b_{51}=0.85	0.0108	-0.0138	-0.0061	0.0080	0.0093	-0.0066	-0.0083	0.0115	-0.0108
b_{61}=0.90	-0.0028	-0.0069	-0.0050	-0.0140	-0.0033	-0.0059	-0.0057	-0.0129	-0.0065
b_{71}=0.95	0.0002	-0.0065	-0.0039	-0.0029	0.0000	-0.0047	-0.0047	-0.0017	-0.0060
b_{81}=1.00	**0.0000**	-0.0091	-0.0054	-0.0008	-0.0006	-0.0069	-0.0069	0.0002	-0.0083
b_{12}=1.00	**0.0000**	0.0064	0.0087	0.0085	0.0055	0.0076	0.0091	0.0064	0.0092
b_{22}=0.60	0.0017	0.0029	0.0050	0.0051	0.0036	0.0038	0.0047	0.0049	0.0043
b_{32}=0.70	0.0024	0.0044	0.0062	0.0067	0.0047	0.0051	0.0062	0.0072	0.0066
b_{42}=0.80	0.0026	0.0052	0.0069	0.0070	0.0054	0.0059	0.0071	0.0051	0.0080
b_{53}=1.00	**0.0000**	0.0015	0.0003	-0.0031	-0.0002	0.0008	0.0009	-0.0021	0.0041
b_{63}=0.20	0.0099	0.0022	0.0023	0.0095	0.0032	0.0022	0.0021	0.0113	0.0026
b_{73}=0.30	0.0086	0.0019	0.0018	0.0076	0.0025	0.0017	0.0016	0.0092	0.0027
b_{83}=0.40	0.0096	0.0031	0.0025	0.0055	0.0035	0.0028	0.0027	0.0075	0.0041
r_f=0.50	**0.0000**	0.0115	0.0019	0.0108	0.0025	0.0017	0.0034	0.0003	0.0186
λ_1=1.00	-0.0206	-0.0182	-0.0101	-0.0353	-0.0142	-0.0081	-0.0098	-0.0304	-0.0257
λ_2=2.00	-0.0934	-0.1156	-0.0310	-0.0994	-0.0097	-0.1217	-0.1196	0.0095	-0.1349
λ_3=3.00	0.0063	0.0268	0.0184	0.0205	0.0009	0.0195	0.0201	0.0206	0.0160
ϕ_{21}=0.00	-0.1082	-0.1095	-0.0261	-0.1010	-0.0112	-0.1167	-0.1103	-0.0018	-0.1194
ϕ_{31}=0.00	-0.0118	0.0184	0.0092	-0.0067	-0.0088	0.0103	0.0121	-0.0133	0.0194
ϕ_{32}=0.00	-0.0009	-0.0049	0.0015	-0.0053	0.0017	-0.0031	-0.0037	-0.0069	0.0002
				$n = 200$					
No of cases converaged	98	94	99	100	99	100	99	99	99
True values of parameters									
b_{11}=0.65	0.0147	0.0381	0.0373	0.1559	0.0156	0.1791	0.1534	-0.0031	0.2166
b_{21}=0.70	0.0157	0.0304	0.0319	0.1008	0.0156	0.1221	0.1052	0.0111	0.1429
b_{31}=0.75	-0.0001	0.0198	0.0210	0.1039	0.0023	0.1281	0.1110	-0.0067	0.1556
b_{41}=0.80	0.0029	0.0267	0.0269	0.1206	0.0050	0.1469	0.1277	-0.0098	0.1771
b_{51}=0.85	0.0184	-0.0234	0.0093	0.0081	0.0160	0.0260	0.0522	0.1352	0.1069
b_{61}=0.90	0.0011	-0.0007	0.0027	0.0038	0.0022	0.0028	0.0034	-0.0119	0.0134
b_{71}=0.95	0.0122	0.0031	0.0105	0.0112	0.0115	0.0135	0.0176	0.0025	0.0317
b_{81}=1.00	**0.0000**	-0.0062	-0.0017	-0.0006	-0.0001	0.0017	0.0117	0.0003	0.0319
b_{12}=1.00	**0.0000**	0.0353	0.0224	0.0187	0.0047	0.0361	0.0403	0.0086	0.0598
b_{22}=0.60	-0.0029	0.0136	0.0055	0.0050	-0.0002	0.0110	0.0120	-0.0175	0.0226
b_{32}=0.70	0.0033	0.0224	0.0126	0.0121	0.0053	0.0197	0.0212	-0.0073	0.0336
b_{42}=0.80	0.0053	0.0270	0.0160	0.0163	0.0077	0.0246	0.0269	-0.0024	0.0414
b_{53}=1.00	**0.0000**	0.0402	0.0087	0.0059	0.0007	0.0413	0.0666	0.0017	0.0833
b_{63}=0.20	-0.0064	0.0131	-0.0129	-0.0170	-0.0081	-0.0152	0.0132	-0.0148	0.0142
b_{73}=0.30	-0.0065	0.0134	-0.0113	-0.0154	-0.0076	-0.0115	0.0129	-0.0148	0.0160
b_{83}=0.40	-0.0004	0.0233	-0.0038	-0.0087	-0.0016	-0.0006	0.0257	-0.0070	0.0307
r_f=0.50	**0.0000**	-0.0036	-0.0004	-0.0018	-0.0004	-0.0006	-0.0153	-0.0015	-0.0128
λ_1=1.00	-0.0673	-0.0653	-0.0367	-0.0240	-0.0616	0.0050	-0.0242	-0.0065	-0.0193
λ_2=2.00	0.0182	-0.0722	-0.0514	-0.1577	0.0061	-0.1549	-0.1564	-0.0163	-0.2412
λ_3=3.00	0.0302	-0.0695	-0.0103	-0.0064	0.0254	-0.1523	-0.2467	-0.2632	-0.3447
ϕ_{21}=0.00	0.0167	-0.0370	-0.0156	-0.1319	0.0147	-0.1683	-0.1578	0.0421	-0.2124
ϕ_{31}=0.00	0.0027	0.0018	0.0122	0.0174	0.0058	-0.0182	-0.0887	-0.1011	-0.1058
ϕ_{32}=0.00	-0.0501	-0.0232	-0.0411	-0.0463	-0.0491	-0.0330	-0.0237	-0.0696	-0.0385

Remarks: The values in boldface are fixed

Table B.4: Simulation result - Bias

Design D	Factor loadings = B1			Factor correlation = $\Phi 2$			$\lambda_1 = 1$ $\lambda_2 = 2$ $\lambda_3 = 0$		
Constraints	Exact	1a	1b	2a	2b	2c	3a	3b	3aR
$n = 2000$									
No of cases converaged	100	100	99	100	100	99	100	100	100
True value of parameters									
b_{11}=0.65	0.1084	-0.0935	-0.0114	-0.0700	-0.0095	-0.0664	-0.0859	-0.0615	-0.0741
b_{21}=0.70	0.0722	-0.0470	-0.0011	-0.0340	-0.0012	-0.0347	-0.0439	-0.0280	-0.0362
b_{31}=0.75	0.0772	-0.0606	-0.0043	-0.0451	-0.0043	-0.0432	-0.0562	-0.0392	-0.0477
b_{41}=0.80	0.0907	-0.0674	-0.0054	-0.0492	-0.0048	-0.0494	-0.0622	-0.0424	-0.0535
b_{51}=0.85	0.0163	0.0066	-0.0045	0.0031	-0.0005	-0.0098	0.0063	0.0127	-0.0089
b_{61}=0.90	-0.0086	-0.0119	0.0011	-0.0110	-0.0013	-0.0027	-0.0122	-0.0085	-0.0201
b_{71}=0.95	0.0005	-0.0030	-0.0002	-0.0009	-0.0005	-0.0046	-0.0026	0.0005	-0.0145
b_{81}=1.00	**0.0000**	-0.0036	-0.0007	-0.0021	-0.0002	-0.0052	-0.0030	-0.0002	-0.0197
b_{12}=1.00	**0.0000**	0.0041	0.0090	0.0089	0.0058	0.0030	0.0026	0.0072	0.0081
b_{22}=0.60	-0.0068	-0.0064	0.0004	-0.0028	-0.0005	-0.0033	-0.0067	-0.0035	-0.0035
b_{32}=0.70	-0.0047	-0.0035	0.0018	0.0008	0.0008	-0.0024	-0.0038	-0.0008	0.0000
b_{42}=0.80	-0.0071	-0.0051	0.0030	-0.0003	0.0015	-0.0020	-0.0055	-0.0027	-0.0009
b_{53}=1.00	**0.0000**	-0.0053	-0.0025	-0.0040	-0.0007	-0.0002	0.0031	-0.0021	0.0034
b_{63}=0.20	0.0244	0.0224	0.0103	0.0212	0.0090	0.0102	0.0185	0.0227	0.0189
b_{73}=0.30	0.0230	0.0234	0.0192	0.0205	0.0145	0.0187	0.0183	0.0218	0.0191
b_{83}=0.40	0.0259	0.0282	0.0225	0.0241	0.0156	0.0217	0.0224	0.0249	0.0238
r_f=0.50	**0.0000**	0.0144	-0.0031	0.0198	-0.0011	-0.0029	0.0147	0.0015	0.0147
λ_1=1.00	0.0203	0.0105	0.0132	0.0007	0.0092	0.0145	0.0091	0.0194	0.0092
λ_2=2.00	-0.1293	0.0618	-0.0128	0.0297	-0.0080	0.0542	0.0592	0.0319	0.0257
λ_3=0.00	-0.0356	-0.0355	-0.0069	-0.0275	-0.0083	-0.0029	-0.0357	-0.0328	-0.0009
ϕ_{21}=0.00	-0.1060	0.0964	0.0086	0.0688	0.0077	0.0659	0.0890	0.0603	0.0747
ϕ_{31}=0.00	-0.0243	-0.0108	-0.0013	-0.0073	-0.0025	0.0080	-0.0088	-0.0185	0.0198
ϕ_{32}=0.00	0.0036	-0.0166	-0.0007	-0.0101	-0.0028	-0.0022	-0.0216	-0.0088	-0.0218
$n = 200$									
No of cases converaged	97	94	98	98	97	96	95	93	95
True values of parameters									
b_{11}=0.65	0.1280	-0.1026	-0.0202	0.2346	-0.0148	0.1680	0.1643	0.2276	0.2111
b_{21}=0.70	0.0727	-0.0379	0.0275	0.1504	0.0247	0.1223	0.1266	0.1288	0.1438
b_{31}=0.75	0.0585	-0.0822	-0.0153	0.1678	0.0015	0.1335	0.1447	0.1497	0.1550
b_{41}=0.80	0.1069	-0.0481	0.0124	0.1870	0.0126	0.1481	0.1581	0.1718	0.1827
b_{51}=0.85	-0.1258	-0.0584	0.0232	0.0248	0.0312	0.0001	-0.0070	0.0178	0.0039
b_{61}=0.90	0.0719	0.0485	0.0263	0.0046	0.0116	0.0198	0.0090	0.0105	0.0163
b_{71}=0.95	0.0276	0.0337	0.0284	0.0121	0.0131	0.0221	0.0151	0.0081	0.0073
b_{81}=1.00	**0.0000**	0.0053	-0.0003	-0.0001	0.0000	-0.0056	-0.0095	0.0001	-0.0075
b_{12}=1.00	**0.0000**	0.0234	0.0231	0.0174	0.0045	0.0396	0.0460	0.0082	0.0543
b_{22}=0.60	-0.0370	-0.0478	-0.0399	-0.0277	-0.0506	-0.0157	-0.0219	-0.0482	-0.0162
b_{32}=0.70	-0.0213	-0.0199	-0.0041	0.0019	-0.0357	0.0011	-0.0042	-0.0343	-0.0037
b_{42}=0.80	-0.0494	-0.0609	-0.0298	-0.0250	-0.0482	-0.0075	-0.0076	-0.0440	-0.0079
b_{53}=1.00	**0.0000**	0.0246	0.0089	0.0061	0.0008	0.0352	0.0514	0.0018	0.0541
b_{63}=0.20	-0.0414	0.0846	-0.0047	0.0080	0.0055	0.0245	0.0390	0.0065	0.0290
b_{73}=0.30	0.0259	0.1458	0.0168	0.0258	0.0279	0.0466	0.0501	0.0253	0.0497
b_{83}=0.40	0.0266	0.1458	0.0456	0.0591	0.0667	0.0714	0.0858	0.0615	0.0742
r_f=0.50	**0.0000**	-0.0035	-0.0027	-0.0080	-0.0023	-0.0026	-0.0256	-0.0005	-0.0179
λ_1=1.00	0.0661	0.1448	0.0875	0.0914	0.0901	0.0635	0.0821	0.0900	0.0999
λ_2=2.00	-0.1711	0.0028	-0.0666	-0.3614	-0.0508	-0.3380	-0.3299	-0.3561	-0.3375
λ_3=0.00	0.0078	-0.1675	-0.1298	-0.1302	-0.1413	-0.0962	-0.0851	-0.1315	-0.0896
ϕ_{21}=0.00	-0.0549	0.1821	0.0517	-0.1954	0.0593	-0.1702	-0.1757	-0.1814	-0.2033
ϕ_{31}=0.00	0.1565	0.0091	-0.0119	-0.0275	-0.0333	-0.0158	-0.0190	-0.0211	0.0119
ϕ_{32}=0.00	-0.0411	-0.0079	-0.0684	-0.0432	-0.0610	-0.0054	0.0035	-0.0484	-0.0542

Remarks: The values in boldface are fixed

Table B.5: Simulation result - Bias

Design E	Factor loadings = B2			Factor correlation = $\Phi 1$			$\lambda_1 = 1$ $\lambda_2 = 2$ $\lambda_3 = 3$		
Constraints	Exact	1a	1b	2a	2b	2c	3a	3b	3aR
				$n = 2000$					
No of cases converaged	96	87	100	94	100	89	89	93	94
True value of parameters									
b_{11}=0.30	-0.0031	-0.0024	-0.0037	-0.0026	-0.0019	-0.0104	-0.0045	-0.0019	-0.0031
b_{21}=0.35	-0.0008	-0.0028	-0.0021	-0.0019	-0.0004	-0.0072	-0.0049	0.0003	-0.0016
b_{31}=0.40	0.0010	-0.0004	-0.0001	0.0013	0.0019	-0.0058	-0.0015	0.0020	0.0006
b_{41}=0.45	-0.0038	-0.0067	-0.0052	-0.0047	-0.0027	-0.0126	-0.0076	-0.0026	-0.0043
b_{51}=0.50	-0.0007	-0.0139	-0.0109	-0.0080	-0.0041	-0.0171	-0.0132	-0.0061	-0.0060
b_{61}=0.55	-0.0049	-0.0121	-0.0096	-0.0081	-0.0061	-0.0137	-0.0123	-0.0066	-0.0101
b_{71}=0.60	0.0014	-0.0048	-0.0036	-0.0018	0.0003	-0.0084	-0.0049	0.0016	-0.0029
b_{81}=0.65	**0.0000**	-0.0092	-0.0064	-0.0049	-0.0012	-0.0105	-0.0087	-0.0025	-0.0055
b_{12}=1.00	**0.0000**	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	-0.0024
b_{22}=0.60	-0.0002	0.0007	0.0000	0.0009	-0.0003	0.0016	0.0016	-0.0007	-0.0011
b_{32}=0.70	-0.0029	-0.0030	-0.0031	-0.0037	-0.0036	-0.0011	-0.0027	-0.0038	-0.0054
b_{42}=0.80	0.0006	0.0011	0.0003	0.0005	-0.0003	0.0024	0.0006	-0.0009	-0.0027
b_{53}=1.00	0.0000	0.0005	-0.0005	-0.0005	-0.0005	0.0001	0.0025	-0.0008	-0.0017
b_{63}=0.20	0.0017	0.0018	0.0010	0.0013	0.0004	0.0034	0.0020	-0.0003	0.0013
b_{73}=0.30	-0.0006	-0.0016	-0.0015	-0.0016	-0.0021	0.0005	-0.0010	-0.0035	-0.0023
b_{83}=0.40	0.0004	-0.0005	-0.0006	-0.0007	-0.0015	0.0009	0.0000	-0.0021	-0.0020
r_f=0.50	**0.0000**	-0.0025	-0.0016	-0.0075	-0.0025	-0.0018	-0.0081	-0.0004	-0.0126
λ_1=1.00	-0.0096	0.0004	-0.0006	0.0081	0.0034	-0.0093	0.0069	0.0032	0.0157
λ_2=2.00	0.0023	0.0018	0.0021	0.0042	0.0001	0.0105	0.0054	-0.0028	0.0166
λ_3=3.00	0.0085	0.0170	0.0183	0.0170	0.0092	0.0291	0.0126	0.0096	0.0213
ϕ_{21}=0.50	0.0005	0.0090	0.0044	0.0019	-0.0008	0.0143	0.0084	-0.0019	0.0085
ϕ_{31}=0.50	0.0040	0.0359	0.0270	0.0207	0.0106	0.0388	0.0343	0.0180	0.0299
ϕ_{32}=0.00	-0.0052	0.0002	-0.0017	-0.0033	-0.0053	0.0037	0.0009	-0.0050	-0.0123
				$n = 200$					
No of cases converaged	94	97	95	94	95	94	99	91	98
True values of parameters									
b_{11}=0.30	0.0222	0.0306	0.0286	0.0281	0.0315	0.0320	0.0298	0.0289	-0.0322
b_{21}=0.35	0.0171	0.0243	0.0216	0.0149	0.0156	0.0233	0.0229	0.0171	-0.0098
b_{31}=0.40	0.0098	0.0182	0.0152	0.0135	0.0117	0.0178	0.0144	0.0163	-0.0213
b_{41}=0.45	0.0045	0.0142	0.0126	0.0062	0.0116	0.0139	0.0061	0.0126	-0.0376
b_{51}=0.50	0.0072	-0.0489	-0.0079	-0.0345	-0.0138	-0.0441	-0.1097	-0.0042	-0.0157
b_{61}=0.55	0.0061	-0.0049	0.0043	0.0031	0.0052	0.0009	-0.0186	0.0048	-0.0106
b_{71}=0.60	0.0114	-0.0021	0.0094	0.0107	0.0042	0.0020	-0.0179	0.0109	-0.0035
b_{81}=0.65	**0.0000**	-0.0121	0.0000	0.0010	0.0002	-0.0084	-0.0213	0.0003	-0.0066
b_{12}=1.00	**0.0000**	0.0121	0.0046	0.0032	0.0008	0.0123	0.0163	0.0016	0.0215
b_{22}=0.60	0.0008	0.0015	0.0002	0.0024	0.0029	0.0022	0.0020	0.0014	0.0037
b_{32}=0.70	0.0027	0.0040	0.0041	0.0003	0.0054	0.0065	0.0040	0.0032	0.0042
b_{42}=0.80	0.0027	0.0030	0.0011	0.0010	0.0011	0.0048	0.0065	-0.0001	0.0092
b_{53}=1.00	**0.0000**	0.0407	0.0089	0.0065	0.0009	0.0413	0.0611	0.0017	0.0498
b_{63}=0.20	0.0066	0.0153	0.0034	0.0006	0.0009	0.0147	0.0205	0.0022	0.0182
b_{73}=0.30	0.0090	0.0184	0.0076	0.0032	0.0077	0.0193	0.0249	0.0046	0.0177
b_{83}=0.40	0.0112	0.0226	0.0090	0.0020	0.0054	0.0256	0.0235	0.0081	0.0179
r_f=0.50	**0.0000**	-0.0206	-0.0026	-0.0110	-0.0027	-0.0036	-0.0565	-0.0007	-0.0475
λ_1=1.00	-0.1910	-0.0856	-0.1535	-0.1072	-0.1665	-0.2085	0.0266	-0.1394	-0.0105
λ_2=2.00	-0.0046	-0.0606	-0.0239	-0.0209	-0.0199	-0.0550	-0.0577	-0.0382	0.0203
λ_3=3.00	0.1414	0.0296	0.1150	0.1512	0.1561	0.0954	-0.0003	0.1116	-0.0154
ϕ_{21}=0.50	-0.0178	-0.0301	-0.0302	-0.0203	-0.0257	-0.0270	-0.0042	-0.0268	0.0907
ϕ_{31}=0.50	-0.0217	0.0442	0.0034	0.0404	0.0132	0.0305	0.1223	-0.0006	0.0992
ϕ_{32}=0.00	-0.0449	0.0005	-0.0340	-0.0357	-0.0402	-0.0088	0.0313	-0.0510	-0.0590

Remarks: The values in boldface are fixed

Table B.6: Simulation result - Bias

Design F	Factor loadings = B2			Factor correlation = Φ1			$\lambda_1 = 1$ $\lambda_2 = 2$ $\lambda_3 = 0$		
Constraints	Exact	1a	1b	2a	2b	2c	3a	3b	3aR
					$n = 2000$				
No of cases converaged	100	100	100	100	100	100	100	100	100
True value of parameters									
b_{11}=0.30	-0.0013	0.0191	-0.0038	-0.0034	-0.0116	0.0108	0.0693	0.0036	-0.0298
b_{21}=0.35	0.0191	0.0538	0.0275	0.0361	0.0137	0.0519	0.1027	0.0286	0.0237
b_{31}=0.40	0.0156	0.0575	0.0389	0.0365	0.0211	0.0655	0.1113	0.0296	0.0211
b_{41}=0.45	0.0167	0.0622	0.0333	0.0381	0.0144	0.0641	0.1220	0.0314	0.0195
b_{51}=0.50	-0.0250	0.0284	0.0072	0.0049	-0.0031	0.0369	0.0468	-0.0201	0.0521
b_{61}=0.55	0.0057	0.0227	0.0135	0.0016	0.0034	0.0290	0.0324	0.0076	0.0393
b_{71}=0.60	0.0085	0.0355	0.0152	0.0111	0.0045	0.0340	0.0481	0.0114	0.0558
b_{81}=0.65	**0.0000**	0.0384	0.0147	0.0111	0.0026	0.0366	0.0536	0.0042	0.0622
b_{12}=1.00	**0.0000**	-0.0002	0.0000	0.0000	0.0000	0.0001	0.0001	0.0000	0.0383
b_{22}=0.60	-0.0174	-0.0463	-0.0299	-0.0399	-0.0223	-0.0193	-0.0878	-0.0257	-0.0418
b_{32}=0.70	-0.0178	-0.0523	-0.0420	-0.0444	-0.0314	-0.0628	-0.0973	-0.0313	-0.0466
b_{42}=0.80	-0.0198	-0.0539	-0.0381	-0.0448	-0.0271	-0.0616	-0.1030	-0.0329	-0.0458
b_{53}=1.00	0.0000	0.0305	0.0176	0.0136	0.0035	0.0330	0.0490	0.0069	0.0559
b_{63}=0.20	-0.0164	-0.0163	-0.0007	0.0049	0.0053	-0.0112	-0.0191	-0.0140	-0.0092
b_{73}=0.30	-0.0078	-0.0118	0.0056	0.0122	0.0112	-0.0068	-0.0158	-0.0053	-0.0032
b_{83}=0.40	-0.0003	-0.0128	0.0071	0.0146	0.0135	-0.0077	-0.0186	0.0018	-0.0068
r_f=0.50	**0.0000**	0.0364	0.0011	0.0308	0.0007	0.0021	0.0363	0.0029	0.0406
λ_1=1.00	0.0474	0.0108	0.0555	0.0370	0.0643	0.0579	0.0259	0.0506	0.0067
λ_2=2.00	0.0036	-0.0415	-0.0025	-0.0217	0.0022	-0.0185	-0.1307	-0.0067	-0.1157
λ_3=0.00	-0.0240	-0.0968	-0.0577	-0.0865	-0.0546	-0.0861	-0.1155	-0.0349	-0.1260
ϕ_{21}=0.50	0.0089	-0.0071	0.0191	0.0282	0.0361	0.0031	-0.0554	0.0047	0.0219
ϕ_{31}=0.50	0.0389	-0.0531	-0.0282	-0.0283	-0.0083	-0.0664	-0.0859	0.0244	-0.0898
ϕ_{32}=0.00	-0.0247	-0.0540	-0.0312	-0.0401	-0.0308	-0.0458	-0.0648	-0.0260	-0.0194
					$n = 200$				
No of cases converaged	95	95	94	94	95	96	95	96	96
True values of parameters									
b_{11}=0.30	0.0390	0.0366	0.0373	0.0360	0.0398	0.0482	0.0408	0.0393	-0.0489
b_{21}=0.35	0.0707	0.1210	0.0590	0.0463	0.0719	0.1381	0.1336	0.0730	0.0736
b_{31}=0.40	0.0859	0.1203	0.0856	0.0689	0.0857	0.1214	0.1049	0.0877	0.0560
b_{41}=0.45	0.0654	0.0959	0.0765	0.0671	0.0676	0.1145	0.1258	0.0685	0.0435
b_{51}=0.50	-0.0098	0.0359	0.0084	-0.0273	-0.0098	0.0511	0.0745	-0.0057	0.1062
b_{61}=0.55	0.0025	-0.0009	0.0052	-0.0028	0.0024	0.0101	-0.0117	0.0030	0.0047
b_{71}=0.60	0.0099	0.0228	0.0121	0.0154	0.0102	0.0277	0.0123	0.0099	0.0333
b_{81}=0.65	**0.0000**	0.0141	0.0040	0.0030	0.0005	0.0138	0.0108	0.0009	0.0336
b_{12}=1.00	**0.0000**	0.0108	0.0041	0.0027	0.0005	0.0140	0.0177	0.0010	0.0465
b_{22}=0.60	-0.0745	-0.1128	-0.0550	-0.0427	-0.0748	-0.1166	-0.1204	-0.0763	-0.0981
b_{32}=0.70	-0.0925	-0.1152	-0.0832	-0.0697	-0.0918	-0.1057	-0.0954	-0.0942	-0.0901
b_{42}=0.80	-0.0789	-0.0894	-0.0774	-0.0671	-0.0798	-0.0990	-0.1197	-0.0821	-0.0869
b_{53}=1.00	**0.0000**	0.0455	0.0166	0.0102	0.0024	0.0542	0.0655	0.0045	0.0750
b_{63}=0.20	0.0416	0.0534	0.0474	0.0402	0.0420	0.0371	0.0494	0.0447	0.0506
b_{73}=0.30	0.0410	0.0477	0.0486	0.0325	0.0415	0.0331	0.0393	0.0430	0.0413
b_{83}=0.40	0.0440	0.0462	0.0495	0.0349	0.0444	0.0433	0.0343	0.0469	0.0366
r_f=0.50	**0.0000**	0.0205	-0.0003	0.0212	0.0002	-0.0007	0.0099	0.0000	0.0155
λ_1=1.00	0.1873	0.1543	0.1866	0.1173	0.1847	0.1868	0.1833	0.1981	0.1618
λ_2=2.00	-0.1094	-0.1205	-0.1052	-0.1154	-0.1130	-0.1161	-0.1323	-0.1146	-0.1166
λ_3=0.00	-0.1453	-0.1813	-0.1601	-0.1097	-0.1427	-0.1823	-0.2124	-0.1584	-0.2376
ϕ_{21}=0.50	0.0340	0.0383	0.0324	0.0344	0.0319	0.0005	0.0528	0.0336	0.1479
ϕ_{31}=0.50	-0.0419	-0.1107	-0.0814	-0.0145	-0.0431	-0.1455	-0.1090	-0.0526	-0.1208
ϕ_{32}=0.00	-0.0709	-0.0845	-0.0786	-0.0670	-0.0697	-0.0854	-0.0691	-0.0732	-0.0893

Remarks: The values in boldface are fixed

Table B.7: Simulation result - Bias

Design G	Factor loadings = B2			Factor correlation = $\Phi 2$			$\lambda_1 = 1$ $\lambda_2 = 2$ $\lambda_3 = 3$		
Constraints	Exact	1a	1b	2a	2b	2c	3a	3b	3aR
					$n = 2000$				
No of cases converaged	100	99	99	100	99	100	96	100	96
True value of parameters									
b_{11}=0.30	0.1203	-0.0669	0.0045	-0.0514	-0.0006	-0.0334	-0.0415	0.1053	-0.0430
b_{21}=0.35	0.0794	-0.0329	0.0061	-0.0253	0.0024	-0.0180	-0.0174	0.0712	-0.0188
b_{31}=0.40	0.0869	-0.0422	0.0042	-0.0329	-0.0003	-0.0234	-0.0250	0.0776	-0.0260
b_{41}=0.45	0.1018	-0.0468	0.0049	-0.0360	0.0001	-0.0267	-0.0256	0.0910	-0.0274
b_{51}=0.50	0.0092	-0.0067	-0.0119	0.0029	0.0032	-0.0076	-0.0018	0.0082	-0.0218
b_{61}=0.55	-0.0155	-0.0184	-0.0051	-0.0157	-0.0022	-0.0055	-0.0188	-0.0153	-0.0233
b_{71}=0.60	-0.0042	-0.0110	-0.0052	-0.0073	-0.0008	-0.0053	-0.0113	-0.0046	-0.0179
b_{81}=0.65	**0.0000**	-0.0047	-0.0061	-0.0011	-0.0006	-0.0065	-0.0052	0.0004	-0.0144
b_{12}=1.00	**0.0000**	0.0089	0.0083	0.0078	0.0054	0.0074	0.0062	0.0065	0.0113
b_{22}=0.60	0.0030	0.0053	0.0041	0.0054	0.0030	0.0037	0.0041	0.0050	0.0074
b_{32}=0.70	0.0048	0.0078	0.0059	0.0075	0.0046	0.0052	0.0077	0.0071	0.0113
b_{42}=0.80	0.0027	0.0076	0.0067	0.0069	0.0051	0.0060	0.0071	0.0054	0.0112
b_{53}=1.00	**0.0000**	-0.0005	0.0019	-0.0019	-0.0002	0.0027	-0.0087	-0.0013	-0.0062
b_{63}=0.20	0.0144	0.0111	0.0014	0.0096	0.0013	0.0018	0.0105	0.0114	0.0111
b_{73}=0.30	0.0130	0.0107	0.0014	0.0091	0.0008	0.0017	0.0098	0.0100	0.0106
b_{83}=0.40	0.0113	0.0067	0.0020	0.0060	0.0015	0.0025	0.0056	0.0077	0.0067
r_f=0.50	**0.0000**	0.0032	0.0015	0.0056	0.0015	0.0013	0.0101	0.0000	0.0104
λ_1=1.00	-0.0666	-0.0512	-0.0104	-0.0493	-0.0118	-0.0103	-0.0670	-0.0487	-0.0674
λ_2=2.00	-0.0947	0.0467	-0.0122	0.0267	-0.0019	0.0145	0.0367	-0.0978	0.0188
λ_3=3.00	0.0230	0.0417	0.0209	0.0246	0.0060	0.0109	0.0565	0.0202	0.0669
ϕ_{21}=0.00	-0.1178	0.0636	-0.0062	0.0503	0.0002	0.0316	0.0381	-0.1057	0.0385
ϕ_{31}=0.00	-0.0156	0.0093	0.0191	-0.0017	0.0021	0.0152	0.0096	-0.0105	0.0371
ϕ_{32}=0.00	-0.0022	-0.0106	0.0031	-0.0078	0.0015	-0.0053	-0.0104	-0.0035	-0.0134
					$n = 200$				
No of cases converaged	99	97	99	99	99	97	97	96	96
True values of parameters									
b_{11}=0.30	-0.0048	0.0017	-0.0134	0.1811	-0.0076	0.1808	0.0119	-0.0134	0.0101
b_{21}=0.35	0.0067	0.0187	0.0091	0.1250	0.0057	0.1340	0.0269	0.0030	0.0210
b_{31}=0.40	-0.0149	-0.0015	-0.0168	0.1245	-0.0162	0.1355	0.0059	-0.0229	-0.0011
b_{41}=0.45	-0.0153	-0.0014	-0.0170	0.1454	-0.0166	0.1535	0.0118	-0.0236	0.0035
b_{51}=0.50	0.0186	-0.0095	-0.0004	-0.0033	0.0173	0.0002	0.0046	0.0003	0.0096
b_{61}=0.55	0.0089	-0.0001	0.0103	0.0083	0.0090	0.0019	0.0013	0.0086	-0.0024
b_{71}=0.60	0.0178	0.0103	0.0177	0.0173	0.0178	0.0133	0.0149	0.0193	0.0090
b_{81}=0.65	**0.0000**	-0.0069	0.0011	0.0010	0.0003	-0.0071	-0.0001	0.0007	-0.0053
b_{12}=1.00	**0.0000**	0.0345	0.0216	0.0175	0.0045	0.0363	0.0391	0.0075	0.0447
b_{22}=0.60	0.0014	0.0098	0.0073	0.0059	0.0027	0.0131	0.0130	0.0021	0.0172
b_{32}=0.70	0.0061	0.0178	0.0140	0.0121	0.0079	0.0192	0.0206	0.0082	0.0262
b_{42}=0.80	0.0108	0.0229	0.0190	0.0163	0.0125	0.0251	0.0262	0.0136	0.0321
b_{53}=1.00	**0.0000**	0.0469	0.0085	0.0053	0.0007	0.0492	0.0648	0.0013	0.0689
b_{63}=0.20	-0.0103	-0.0139	-0.0179	-0.0165	-0.0119	-0.0126	-0.0090	-0.0170	-0.0130
b_{73}=0.30	-0.0096	-0.0137	-0.0140	-0.0133	-0.0113	-0.0134	-0.0133	-0.0187	-0.0151
b_{83}=0.40	-0.0026	-0.0046	-0.0061	-0.0084	-0.0042	-0.0010	-0.0003	-0.0109	-0.0040
r_f=0.50	**0.0000**	-0.0060	-0.0012	-0.0023	-0.0005	-0.0022	-0.0061	-0.0003	-0.0026
λ_1=1.00	-0.0570	0.0190	-0.0016	0.0101	-0.0450	-0.0152	0.0092	-0.0206	0.0280
λ_2=2.00	0.0209	-0.0592	-0.0187	-0.1659	0.0124	-0.1183	-0.0360	0.0024	-0.0575
λ_3=3.00	0.0339	-0.1349	-0.0268	-0.0137	0.0271	-0.1546	-0.1778	0.0334	-0.2030
ϕ_{21}=0.00	0.0450	0.0107	0.0371	-0.1625	0.0449	-0.1754	0.0029	0.0519	0.0388
ϕ_{31}=0.00	0.0283	0.0259	0.0367	0.0430	0.0304	0.0082	-0.0022	0.0563	0.0704
ϕ_{32}=0.00	-0.0498	-0.0171	-0.0353	-0.0452	-0.0483	-0.0295	-0.0172	-0.0526	-0.0161

Remarks: The values in boldface are fixed

Table B.8: Simulation result - Bias

Design H	Factor loadings = B2			Factor correlation = $\Phi 2$			$\lambda_1 = 1$ $\lambda_2 = 2$ $\lambda_3 = 0$		
Constraints	Exact	1a	1b	2a	2b	2c	3a	3b	3aR
$n = 2000$									
No of cases converaged	100	99	98	100	100	100	100	100	99
True value of parameters									
b_{11}=0.30	0.0793	0.1144	-0.0036	0.1190	-0.0037	0.1042	0.1208	0.0872	0.1787
b_{21}=0.35	0.0579	0.0869	0.0116	0.0909	0.0060	0.0805	0.1099	0.0652	0.1383
b_{31}=0.40	0.0647	0.0997	0.0115	0.1066	0.0045	0.0922	0.1283	0.0733	0.1616
b_{41}=0.45	0.0763	0.1163	0.0152	0.1252	0.0061	0.1064	0.1473	0.0865	0.1870
b_{51}=0.50	0.0087	0.0105	0.0016	0.0273	-0.0091	0.0062	0.0386	0.0100	0.0659
b_{61}=0.55	-0.0085	-0.0134	0.0007	-0.0134	0.0014	-0.0018	-0.0174	-0.0093	-0.0095
b_{71}=0.60	0.0008	-0.0028	0.0005	-0.0004	0.0002	-0.0015	-0.0051	0.0009	0.0042
b_{81}=0.65	**0.0000**	-0.0021	0.0017	0.0029	0.0001	0.0019	0.0024	0.0005	0.0117
b_{12}=1.00	**0.0000**	0.0133	0.0107	0.0123	0.0062	0.0170	0.0314	0.0084	0.0377
b_{22}=0.60	-0.0114	-0.0237	-0.0131	-0.0327	-0.0049	-0.0205	-0.0638	-0.0096	-0.0115
b_{32}=0.70	-0.0141	-0.0279	-0.0143	-0.0404	-0.0050	-0.0238	-0.0768	-0.0123	-0.0497
b_{42}=0.80	-0.0182	-0.0340	-0.0181	-0.0492	-0.0061	-0.0283	-0.0875	-0.0158	-0.0586
b_{53}=1.00	0.0000	0.0088	-0.0024	-0.0053	-0.0004	0.0052	0.0141	-0.0013	0.0244
b_{63}=0.20	0.0182	0.0379	0.0216	0.0501	0.0052	0.0290	0.0767	0.0230	0.0625
b_{73}=0.30	0.0175	0.0420	0.0321	0.0546	0.0125	0.0394	0.0849	0.0234	0.0701
b_{83}=0.40	0.0220	0.0509	0.0372	0.0634	0.0148	0.0440	0.0949	0.0288	0.0786
r_f=0.50	0.0000	0.0036	-0.0055	0.0045	-0.0028	-0.0054	-0.0048	0.0021	-0.0032
λ_1=1.00	0.0550	0.1119	0.0727	0.1433	0.0327	0.1014	0.2374	0.0602	0.1988
λ_2=2.00	-0.1085	-0.2044	-0.0382	-0.2275	-0.0178	-0.1712	-0.2695	-0.1381	-0.3190
λ_3=0.00	-0.0444	-0.0965	-0.0483	-0.1359	-0.0116	-0.0755	-0.2037	-0.0523	-0.1937
ϕ_{21}=0.00	-0.0688	-0.0898	0.0159	-0.0837	0.0069	-0.0852	-0.0599	-0.0802	-0.1207
ϕ_{31}=0.00	-0.0078	-0.0244	-0.0124	-0.0510	0.0126	-0.0226	-0.0913	-0.0136	-0.0958
ϕ_{32}=0.00	-0.0050	-0.0059	-0.0096	-0.0019	-0.0090	-0.0047	-0.0171	0.0017	-0.0074
$n = 200$									
No of cases converaged	96	93	92	94	93	90	91	96	91
True values of parameters									
b_{11}=0.30	0.0332	-0.0045	0.0171	0.1723	0.0588	0.2134	-0.0368	0.0333	-0.0210
b_{21}=0.35	0.0444	0.0287	0.0420	0.1294	0.0621	0.1589	0.0162	0.0518	0.0595
b_{31}=0.40	0.0233	0.0079	0.0213	0.1185	0.0444	0.1737	-0.0110	0.0230	0.0451
b_{41}=0.45	0.0703	0.0413	0.0530	0.1419	0.0597	0.1951	0.0229	0.0581	0.0653
b_{51}=0.50	-0.0215	0.0999	0.0804	0.0569	0.0356	0.0531	0.0453	0.0228	0.0840
b_{61}=0.55	0.0167	-0.0174	-0.0120	-0.0004	0.0479	-0.0151	-0.0016	0.0142	0.0439
b_{71}=0.60	0.0455	0.0239	0.0329	0.0146	0.0549	0.0194	0.0469	0.0547	0.0773
b_{81}=0.65	**0.0000**	0.0004	-0.0008	0.0006	0.0001	-0.0052	0.0086	0.0003	0.0187
b_{12}=1.00	**0.0000**	0.0408	0.0225	0.0183	0.0044	0.0408	0.0484	0.0076	0.0580
b_{22}=0.60	-0.0603	-0.0314	-0.0460	-0.0391	-0.0578	-0.0516	-0.0161	-0.0588	-0.0520
b_{32}=0.70	-0.0487	-0.0183	-0.0293	-0.0202	-0.0454	-0.0380	0.0025	-0.0384	-0.0441
b_{42}=0.80	-0.0874	-0.0443	-0.0546	-0.0422	-0.0551	-0.0762	-0.0257	-0.0709	-0.0531
b_{53}=1.00	**0.0000**	0.0287	0.0095	0.0055	0.0009	0.0248	0.0437	0.0016	0.0573
b_{63}=0.20	-0.0025	-0.0137	-0.0037	-0.0017	-0.0592	0.0451	-0.0046	0.0080	-0.0337
b_{73}=0.30	-0.0217	-0.0407	-0.0288	-0.0137	-0.0491	0.0281	-0.0362	-0.0110	-0.0484
b_{83}=0.40	0.0360	0.0135	0.0235	0.0176	0.0298	0.0654	0.0134	0.0494	0.0320
r_f=0.50	**0.0000**	-0.0073	-0.0033	-0.0039	-0.0037	-0.0022	-0.0193	-0.0007	-0.0155
λ_1=1.00	0.1534	0.1437	0.1443	0.0995	0.1383	0.2035	0.1333	0.2113	0.1557
λ_2=2.00	-0.1232	-0.1135	-0.1199	-0.3064	-0.1471	-0.4265	-0.0698	-0.1302	-0.1132
λ_3=0.00	-0.1180	-0.1996	-0.1909	-0.1299	-0.1541	-0.2169	-0.1496	-0.2044	-0.1953
ϕ_{21}=0.00	0.0365	0.0488	0.0246	-0.1345	0.0007	-0.1412	0.0504	0.0550	0.0473
ϕ_{31}=0.00	0.0535	-0.0434	-0.0234	-0.0038	-0.0024	-0.0539	-0.0132	-0.0051	-0.0442
ϕ_{32}=0.00	-0.0614	-0.0565	-0.0470	-0.0311	-0.0446	-0.0297	-0.0416	-0.0706	-0.0570

Remarks: The values in boldface are fixed

Appendix C

Simulation result - RMSE

Table C.1: Simulation result - RMSE

Design A	Factor loadings = B1			Factor correlation = $\Phi 1$			$\lambda_1 = 1$ $\lambda_2 = 2$ $\lambda_3 = 3$		
Constraints	Exact	1a	1b	2a	2b	2c	3a	3b	3aR
$n = 2000$									
No of cases converaged	98	97	99	94	100	92	95	95	99
True value of parameters									
b_{11}=0.65	0.0612	0.0697	0.0659	0.0657	0.0620	0.0699	0.0704	0.0636	0.0618
b_{21}=0.70	0.0449	0.0518	0.0491	0.0489	0.0458	0.0522	0.0520	0.0461	0.0441
b_{31}=0.75	0.0500	0.0570	0.0537	0.0538	0.0506	0.0570	0.0577	0.0517	0.0492
b_{41}=0.80	0.0564	0.0638	0.0606	0.0592	0.0570	0.0643	0.0645	0.0580	0.0561
b_{51}=0.85	0.0620	0.0931	0.0845	0.0811	0.0660	0.0951	0.0954	0.0676	0.0925
b_{61}=0.90	0.0231	0.0371	0.0324	0.0310	0.0248	0.0375	0.0381	0.0268	0.0366
b_{71}=0.95	0.0280	0.0452	0.0398	0.0382	0.0306	0.0458	0.0464	0.0324	0.0447
b_{81}=1.00	**0.0000**	0.0404	0.0314	0.0281	0.0092	0.0410	0.0424	0.0153	0.0401
b_{12}=1.00	**0.0000**	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0258
b_{22}=0.60	0.0193	0.0204	0.0198	0.0202	0.0195	0.0197	0.0207	0.0199	0.0218
b_{32}=0.70	0.0196	0.0204	0.0200	0.0201	0.0197	0.0201	0.0206	0.0202	0.0227
b_{42}=0.80	0.0202	0.0214	0.0204	0.0207	0.0201	0.0207	0.0230	0.0202	0.0256
b_{53}=1.00	**0.0000**	0.0314	0.0234	0.0207	0.0068	0.0318	0.0332	0.0113	0.0372
b_{63}=0.20	0.0299	0.0291	0.0293	0.0295	0.0297	0.0295	0.0293	0.0302	0.0293
b_{73}=0.30	0.0298	0.0292	0.0289	0.0295	0.0295	0.0291	0.0297	0.0297	0.0295
b_{83}=0.40	0.0284	0.0275	0.0275	0.0277	0.0281	0.0279	0.0280	0.0277	0.0287
r_f=0.50	**0.0000**	0.1079	0.0260	0.1474	0.0274	0.0214	0.2011	0.0059	0.1997
λ_1=1.00	0.1025	0.1563	0.1073	0.1838	0.1077	0.1067	0.2412	0.1051	0.2395
λ_2=2.00	0.1111	0.1232	0.1138	0.1267	0.1116	0.1175	0.1355	0.1137	0.1331
λ_3=3.00	0.0911	0.2074	0.1731	0.1650	0.1073	0.2067	0.2120	0.1222	0.2246
ϕ_{21}=0.50	0.0733	0.0872	0.0819	0.0808	0.0753	0.0877	0.0878	0.0772	0.0716
ϕ_{31}=0.50	0.0994	0.1548	0.1386	0.1340	0.1063	0.1552	0.1566	0.1087	0.1587
ϕ_{32}=0.00	0.0401	0.0392	0.0391	0.0396	0.0396	0.0389	0.0394	0.0402	0.0501
$n = 200$									
No of cases converaged	97	96	98	98	98	99	97	98	98
True values of parameters									
b_{11}=0.65	0.2192	0.2303	0.2146	0.2165	0.2299	0.2849	0.2331	0.2177	0.2912
b_{21}=0.70	0.1608	0.1675	0.1566	0.1621	0.1645	0.2006	0.1742	0.1614	0.2008
b_{31}=0.75	0.1800	0.1871	0.1760	0.1792	0.1821	0.2197	0.1950	0.1795	0.2136
b_{41}=0.80	0.2139	0.2196	0.2085	0.2129	0.2171	0.2451	0.2297	0.2124	0.2425
b_{51}=0.85	0.2644	0.2654	0.2592	0.2634	0.2641	0.2972	0.2833	0.2623	0.3734
b_{61}=0.90	0.0925	0.1136	0.0942	0.0919	0.0922	0.1178	0.1243	0.0925	0.1305
b_{71}=0.95	0.0874	0.1146	0.0908	0.0927	0.0877	0.1260	0.1389	0.0878	0.1453
b_{81}=1.00	**0.0000**	0.0958	0.0338	0.0225	0.0039	0.1034	0.1342	0.0076	0.1402
b_{12}=1.00	**0.0000**	0.0288	0.0156	0.0125	0.0029	0.0301	0.0362	0.0053	0.1003
b_{22}=0.60	0.0652	0.0662	0.0637	0.0671	0.0652	0.0656	0.0672	0.0652	0.0829
b_{32}=0.70	0.0674	0.0683	0.0654	0.0677	0.0669	0.0679	0.0719	0.0671	0.0881
b_{42}=0.80	0.0693	0.0717	0.0668	0.0703	0.0698	0.0707	0.0745	0.0696	0.0951
b_{53}=1.00	**0.0000**	0.0974	0.0280	0.0194	0.0033	0.1097	0.1652	0.0064	0.1708
b_{63}=0.20	0.1072	0.1179	0.1034	0.1104	0.1068	0.1172	0.1265	0.1070	0.1257
b_{73}=0.30	0.1016	0.1100	0.0964	0.1059	0.1006	0.1108	0.1268	0.1010	0.1167
b_{83}=0.40	0.0980	0.1081	0.0945	0.0985	0.0973	0.1103	0.1310	0.0973	0.1192
r_f=0.50	**0.0000**	0.0732	0.0128	0.1095	0.0130	0.0110	0.2409	0.0026	0.2399
λ_1=1.00	0.5477	0.4249	0.4389	0.5429	0.5410	0.4173	0.4826	0.5386	0.4778
λ_2=2.00	0.4497	0.4571	0.4314	0.4494	0.4776	0.4534	0.4771	0.4477	0.4539
λ_3=3.00	0.5951	0.5395	0.4701	0.5910	0.5902	0.5357	0.6361	0.5855	0.8336
ϕ_{21}=0.50	0.2626	0.2716	0.2520	0.2629	0.2757	0.3309	0.2989	0.2631	0.3298
ϕ_{31}=0.50	0.3840	0.4295	0.3844	0.3820	0.3820	0.4540	0.4910	0.3826	0.5691
ϕ_{32}=0.00	0.1829	0.1559	0.1600	0.1730	0.2451	0.2315	0.1559	0.1788	0.2654

Remarks: The values in boldface are fixed

Table C.2: Simulation result - RMSE

Design B	Factor loadings = B1		Factor correlation = $\Phi 1$			$\lambda_1 = 1$ $\lambda_2 = 2$ $\lambda_3 = 0$			
Constraints	Exact	1a	1b	2a	2b	2c	3a	3b	3aR
					$n = 2000$				
No of cases converaged	100	100	100	100	100	100	100	100	100
True value of parameters									
b_{11}=0.65	0.0849	0.1124	0.0917	0.0875	0.0859	0.1105	0.1383	0.0875	0.1264
b_{21}=0.70	0.0486	0.1102	0.0580	0.0581	0.0501	0.1053	0.1270	0.0496	0.1238
b_{31}=0.75	0.0525	0.1190	0.0636	0.0723	0.0519	0.1077	0.1402	0.0549	0.1345
b_{41}=0.80	0.0611	0.1191	0.0659	0.0731	0.0574	0.1067	0.1447	0.0615	0.1357
b_{51}=0.85	0.1422	0.1473	0.1316	0.1351	0.1289	0.1384	0.1553	0.1349	0.1757
b_{61}=0.90	0.0497	0.0781	0.0545	0.0541	0.0445	0.0749	0.0897	0.0495	0.1025
b_{71}=0.95	0.0381	0.0668	0.0434	0.0384	0.0313	0.0648	0.0795	0.0385	0.0879
b_{81}=1.00	**0.0000**	0.0630	0.0316	0.0253	0.0058	0.0607	0.0760	0.0104	0.0805
b_{12}=1.00	**0.0000**	0.0011	0.0000	0.0000	0.0000	0.0008	0.0050	0.0000	0.0510
b_{22}=0.60	0.0346	0.1302	0.0628	0.0678	0.0580	0.1219	0.1476	0.0409	0.1529
b_{32}=0.70	0.0335	0.1365	0.0650	0.0827	0.0562	0.1189	0.1600	0.0431	0.1652
b_{42}=0.80	0.0305	0.1344	0.0562	0.0763	0.0499	0.1152	0.1619	0.0379	0.1671
b_{53}=1.00	**0.0000**	0.0550	0.0283	0.0236	0.0056	0.0519	0.0701	0.0097	0.0756
b_{63}=0.20	0.1123	0.1496	0.1195	0.1229	0.1128	0.1417	0.1614	0.1135	0.1732
b_{73}=0.30	0.1059	0.1453	0.1148	0.1163	0.1078	0.1369	0.1575	0.1079	0.1659
b_{83}=0.40	0.0895	0.1437	0.1075	0.1100	0.0991	0.1346	0.1570	0.0943	0.1610
r_f=0.50	**0.0000**	0.1246	0.0337	0.1460	0.0357	0.0314	0.1841	0.0078	0.1820
λ_1=1.00	0.0886	0.2333	0.1270	0.2142	0.1314	0.1624	0.3019	0.1036	0.3007
λ_2=2.00	0.0636	0.0865	0.0635	0.0787	0.0614	0.0712	0.1924	0.0634	0.1906
λ_3=0.00	0.2067	0.2685	0.2311	0.2555	0.2382	0.2488	0.2853	0.2138	0.3198
ϕ_{21}=0.50	0.1218	0.1854	0.1495	0.1457	0.1450	0.1816	0.2001	0.1358	0.1592
ϕ_{31}=0.50	0.2213	0.2364	0.2150	0.2222	0.2091	0.2245	0.2459	0.2171	0.2637
ϕ_{32}=0.00	0.0500	0.0793	0.0611	0.0639	0.0645	0.0767	0.0816	0.0565	0.0827
					$n = 200$				
No of cases converaged	99	100	100	99	98	98	95	100	100
True values of parameters									
b_{11}=0.65	0.2389	0.2997	0.2792	0.3088	0.2475	0.3234	0.3687	0.2657	0.3043
b_{21}=0.70	0.2041	0.2570	0.2349	0.2283	0.2125	0.2633	0.3068	0.2228	0.2304
b_{31}=0.75	0.1943	0.2561	0.2180	0.2360	0.1973	0.2494	0.3080	0.2241	0.2276
b_{41}=0.80	0.2517	0.2905	0.2965	0.3075	0.2743	0.3235	0.3598	0.2824	0.3038
b_{51}=0.85	0.4257	0.3035	0.2642	0.3075	0.3979	0.2874	0.3635	0.4171	0.4197
b_{61}=0.90	0.0972	0.1252	0.1009	0.1283	0.0977	0.1270	0.1505	0.1006	0.1744
b_{71}=0.95	0.1052	0.1384	0.1060	0.1514	0.1039	0.1431	0.1467	0.1060	0.1746
b_{81}=1.00	**0.0000**	0.0730	0.0172	0.0109	0.0020	0.0787	0.1155	0.0038	0.1422
b_{12}=1.00	**0.0000**	0.0345	0.0111	0.0090	0.0017	0.0369	0.0955	0.0031	0.1066
b_{22}=0.60	0.1724	0.2363	0.2078	0.1891	0.1881	0.2274	0.2750	0.2136	0.1950
b_{32}=0.70	0.1529	0.2379	0.1817	0.1859	0.1675	0.1984	0.2547	0.2079	0.1712
b_{42}=0.80	0.1581	0.2119	0.2132	0.2158	0.1996	0.2171	0.2735	0.2287	0.2032
b_{53}=1.00	**0.0000**	0.0965	0.0328	0.0245	0.0042	0.1017	0.1444	0.0082	0.1700
b_{63}=0.20	0.2026	0.2331	0.2110	0.2246	0.2022	0.2459	0.2684	0.2125	0.2553
b_{73}=0.30	0.2107	0.2551	0.2179	0.2555	0.2087	0.2678	0.2859	0.2155	0.2629
b_{83}=0.40	0.1768	0.2194	0.1880	0.1952	0.1803	0.2329	0.2541	0.1897	0.2421
r_f=0.50	**0.0000**	0.1160	0.0188	0.1485	0.0195	0.0164	0.2547	0.0039	0.2787
λ_1=1.00	0.2726	0.3486	0.2947	0.3291	0.2854	0.3087	0.5613	0.2822	0.4147
λ_2=2.00	0.3377	0.4651	0.4308	0.4906	0.3456	0.4612	0.4976	0.4361	0.4236
λ_3=0.00	0.5820	0.5186	0.5122	0.4979	0.5676	0.4989	0.6479	0.5723	0.4997
ϕ_{21}=0.50	0.3128	0.3813	0.3475	0.3606	0.3261	0.4188	0.4227	0.3119	0.4291
ϕ_{31}=0.50	0.5682	0.4917	0.4279	0.4559	0.5429	0.4430	0.5622	0.5597	0.6102
ϕ_{32}=0.00	0.2057	0.2253	0.2061	0.2005	0.2089	0.2344	0.3633	0.1987	0.2255

Remarks: The values in boldface are fixed

Table C.3: Simulation result - RMSE

Design C	Factor loadings = B1		Factor correlation = $\Phi 2$			$\lambda_1 = 1$ $\lambda_2 = 2$ $\lambda_3 = 3$			
Constraints	Exact	1a	1b	2a	2b	2c	3a	3b	3aR
$n = 2000$									
No of cases converaged	100	100	100	92	99	100	100	100	100
True value of parameters									
b_{11}=0.65	0.1692	0.1433	0.1470	0.1662	0.1464	0.1513	0.1679	0.1521	0.1591
b_{21}=0.70	0.1030	0.0864	0.0902	0.1018	0.0895	0.0925	0.1013	0.0980	0.0985
b_{31}=0.75	0.1193	0.1004	0.1053	0.1199	0.1046	0.1077	0.1181	0.1107	0.1132
b_{41}=0.80	0.1349	0.1152	0.1188	0.1328	0.1202	0.1215	0.1337	0.1264	0.1276
b_{51}=0.85	0.0567	0.0966	0.0762	0.0715	0.0574	0.0965	0.0839	0.0592	0.1076
b_{61}=0.90	0.0236	0.0280	0.0254	0.0248	0.0235	0.0273	0.0263	0.0236	0.0313
b_{71}=0.95	0.0233	0.0309	0.0262	0.0256	0.0235	0.0305	0.0278	0.0237	0.0347
b_{81}=1.00	**0.0000**	0.0281	0.0166	0.0125	0.0022	0.0301	0.0213	0.0044	0.0367
b_{12}=1.00	**0.0000**	0.0212	0.0169	0.0162	0.0102	0.0207	0.0175	0.0135	0.0281
b_{22}=0.60	0.0178	0.0203	0.0196	0.0195	0.0187	0.0201	0.0198	0.0190	0.0235
b_{32}=0.70	0.0167	0.0203	0.0192	0.0191	0.0181	0.0201	0.0195	0.0185	0.0233
b_{42}=0.80	0.0172	0.0224	0.0209	0.0214	0.0192	0.0221	0.0211	0.0202	0.0272
b_{53}=1.00	**0.0000**	0.0616	0.0435	0.0380	0.0100	0.0628	0.0506	0.0179	0.0677
b_{63}=0.20	0.0507	0.0305	0.0295	0.0278	0.0273	0.0308	0.0296	0.0276	0.0312
b_{73}=0.30	0.0489	0.0326	0.0306	0.0277	0.0275	0.0332	0.0312	0.0278	0.0333
b_{83}=0.40	0.0493	0.0360	0.0323	0.0305	0.0271	0.0370	0.0338	0.0278	0.0372
r_f=0.50	**0.0000**	0.1155	0.0254	0.1509	0.0262	0.0221	0.0408	0.0055	0.2169
λ_1=1.00	0.1136	0.1699	0.1167	0.1951	0.1172	0.1140	0.1208	0.1121	0.2649
λ_2=2.00	0.1752	0.1619	0.1406	0.1822	0.1364	0.1601	0.1786	0.1575	0.1945
λ_3=3.00	0.1136	0.2410	0.1914	0.1731	0.1202	0.2390	0.2112	0.1298	0.2945
ϕ_{21}=0.00	0.1594	0.1390	0.1577	0.1550	0.1581	0.1465	0.1576	0.1535	0.1571
ϕ_{31}=0.00	0.0697	0.1318	0.1039	0.0943	0.0722	0.1236	0.1155	0.0762	0.1545
ϕ_{32}=0.00	0.0445	0.0730	0.0551	0.0535	0.0428	0.0696	0.0660	0.0442	0.0707
$n = 200$									
No of cases converaged	98	94	99	100	99	100	99	99	99
True values of parameters									
b_{11}=0.65	0.2678	0.3003	0.2712	0.3388	0.2695	0.3772	0.3783	0.2965	0.4269
b_{21}=0.70	0.1835	0.2210	0.1927	0.2344	0.1875	0.2564	0.2663	0.2148	0.2887
b_{31}=0.75	0.2019	0.2380	0.2071	0.2535	0.2028	0.2828	0.2927	0.2345	0.3179
b_{41}=0.80	0.2318	0.2581	0.2347	0.2754	0.2315	0.3155	0.3199	0.2643	0.3539
b_{51}=0.85	0.2707	0.3386	0.2777	0.2749	0.2705	0.3353	0.3955	0.3827	0.4158
b_{61}=0.90	0.0940	0.1077	0.0957	0.0958	0.0943	0.1068	0.1235	0.1087	0.1232
b_{71}=0.95	0.0876	0.1077	0.0903	0.0893	0.0877	0.1130	0.1352	0.0881	0.1368
b_{81}=1.00	**0.0000**	0.0599	0.0175	0.0126	0.0021	0.0719	0.1067	0.0048	0.1126
b_{12}=1.00	**0.0000**	0.0587	0.0372	0.0313	0.0078	0.0587	0.0674	0.0140	0.0954
b_{22}=0.60	0.0637	0.0678	0.0639	0.0641	0.0642	0.0647	0.0690	0.1512	0.0772
b_{32}=0.70	0.0664	0.0741	0.0647	0.0656	0.0655	0.0665	0.0738	0.1275	0.0828
b_{42}=0.80	0.0637	0.0806	0.0643	0.0683	0.0633	0.0708	0.0805	0.1264	0.0948
b_{53}=1.00	**0.0000**	0.1260	0.0361	0.0263	0.0041	0.1316	0.1903	0.0083	0.1852
b_{63}=0.20	0.1331	0.2174	0.1348	0.1415	0.1329	0.1520	0.2175	0.1751	0.2204
b_{73}=0.30	0.1293	0.1996	0.1313	0.1368	0.1295	0.1506	0.2025	0.1759	0.2053
b_{83}=0.40	0.1195	0.2047	0.1205	0.1266	0.1192	0.1466	0.2082	0.1779	0.2120
r_f=0.50	**0.0000**	0.0539	0.0096	0.0753	0.0098	0.0080	0.1546	0.0085	0.1539
λ_1=1.00	0.6968	0.6806	0.6630	0.6893	0.6913	0.5379	0.6693	0.7427	0.6600
λ_2=2.00	0.4377	0.4034	0.4354	0.5013	0.4348	0.5986	0.4457	0.4360	0.5304
λ_3=3.00	0.7653	0.7655	0.7626	0.7578	0.7621	0.8290	0.9090	1.1917	0.9922
ϕ_{21}=0.00	0.3272	0.3107	0.3248	0.3235	0.3289	0.3079	0.2993	0.3463	0.3578
ϕ_{31}=0.00	0.3286	0.3794	0.3351	0.3363	0.3283	0.3696	0.4015	0.3370	0.4391
ϕ_{32}=0.00	0.1716	0.1946	0.1679	0.1800	0.1708	0.2265	0.3009	0.1901	0.4393

Remarks: The values in boldface are fixed

Table C.4: Simulation result - RMSE

Design D	Factor loadings = B1		Factor correlation = $\Phi 2$			$\lambda_1 = 1$ $\lambda_2 = 2$ $\lambda_3 = 0$		
Constraints	Exact	1a	1b	2a	2b	2c	3a	3b	3aR
				$n = 2000$					
No of cases converaged	100	100	99	100	100	99	100	100	100
True value of parameters									
b_{11}=0.65	0.1618	0.1168	0.1428	0.1412	0.1391	0.1074	0.1071	0.1386	0.1490
b_{21}=0.70	0.0995	0.0677	0.0829	0.0845	0.0816	0.0630	0.0631	0.0816	0.0890
b_{31}=0.75	0.1159	0.0795	0.0955	0.0994	0.0950	0.0719	0.0744	0.0970	0.1069
b_{41}=0.80	0.1314	0.0910	0.1130	0.1103	0.1094	0.0824	0.0817	0.1075	0.1222
b_{51}=0.85	0.0719	0.1127	0.1013	0.1000	0.0729	0.1120	0.1174	0.0969	0.1181
b_{61}=0.90	0.0209	0.0225	0.0331	0.0357	0.0203	0.0227	0.0236	0.0372	0.0323
b_{71}=0.95	0.0160	0.0301	0.0193	0.0197	0.0160	0.0299	0.0335	0.0200	0.0410
b_{81}=1.00	**0.0000**	0.0376	0.0123	0.0100	0.0021	0.0369	0.0430	0.0035	0.0544
b_{12}=1.00	**0.0000**	0.0213	0.0171	0.0170	0.0112	0.0221	0.0225	0.0144	0.0370
b_{22}=0.60	0.0173	0.0214	0.0191	0.0194	0.0179	0.0211	0.0227	0.0187	0.0286
b_{32}=0.70	0.0165	0.0211	0.0190	0.0188	0.0177	0.0217	0.0220	0.0191	0.0319
b_{42}=0.80	0.0161	0.0223	0.0193	0.0195	0.0178	0.0225	0.0235	0.0192	0.0369
b_{53}=1.00	**0.0000**	0.0889	0.0436	0.0349	0.0074	0.0909	0.1131	0.0136	0.1104
b_{63}=0.20	0.0941	0.0978	0.1033	0.1043	0.0919	0.0893	0.0870	0.1074	0.0866
b_{73}=0.30	0.1014	0.0996	0.1013	0.1021	0.0980	0.0912	0.0898	0.1044	0.0895
b_{83}=0.40	0.1104	0.1079	0.1081	0.1092	0.1068	0.1009	0.1018	0.1100	0.1012
r_f=0.50	**0.0000**	0.1113	0.0330	0.1306	0.0336	0.0298	0.1657	0.0077	0.1656
λ_1=1.00	0.0441	0.1188	0.0492	0.1363	0.0500	0.0490	0.1752	0.0405	0.1750
λ_2=2.00	0.1776	0.1285	0.1419	0.1474	0.1383	0.1124	0.1360	0.1403	0.1287
λ_3=0.00	0.1024	0.1471	0.1240	0.1209	0.0991	0.1441	0.1594	0.1201	0.1470
ϕ_{21}=0.00	0.1515	0.1319	0.1552	0.1496	0.1519	0.1221	0.1192	0.1472	0.1436
ϕ_{31}=0.00	0.1322	0.1743	0.1651	0.1646	0.1336	0.1742	0.1778	0.1605	0.1851
ϕ_{32}=0.00	0.0769	0.0683	0.0711	0.0590	0.0689	0.0681	0.0724	0.0591	0.0730
				$n = 200$					
No of cases converaged	97	94	98	98	97	96	95	93	95
True values of parameters									
b_{11}=0.65	0.3472	0.3126	0.2901	0.4407	0.2912	0.4785	0.5077	0.4242	0.3990
b_{21}=0.70	0.2181	0.2604	0.2648	0.2707	0.2129	0.2700	0.2922	0.2507	0.2471
b_{31}=0.75	0.2456	0.2920	0.3174	0.3050	0.2214	0.3129	0.3488	0.2745	0.2852
b_{41}=0.80	0.2587	0.2957	0.2512	0.3338	0.2370	0.3337	0.3653	0.3025	0.3129
b_{51}=0.85	0.3183	0.2527	0.2961	0.2896	0.2826	0.3137	0.2735	0.2509	0.2672
b_{61}=0.90	0.4719	0.1672	0.1803	0.1796	0.2005	0.1826	0.1347	0.2044	0.1521
b_{71}=0.95	0.3500	0.1159	0.1338	0.1518	0.1482	0.1486	0.1465	0.1312	0.1410
b_{81}=1.00	**0.0000**	0.0499	0.0120	0.0088	0.0015	0.0497	0.0765	0.0031	0.1046
b_{12}=1.00	**0.0000**	0.0626	0.0397	0.0307	0.0081	0.0651	0.0768	0.0145	0.0994
b_{22}=0.60	0.1832	0.2919	0.2434	0.1168	0.1892	0.1220	0.1713	0.1937	0.1203
b_{32}=0.70	0.1605	0.3055	0.2485	0.2363	0.1625	0.1178	0.1808	0.1657	0.1256
b_{42}=0.80	0.1742	0.2901	0.2005	0.1430	0.1776	0.1431	0.1702	0.1813	0.1489
b_{53}=1.00	**0.0000**	0.1274	0.0298	0.0215	0.0037	0.1347	0.2025	0.0073	0.2065
b_{63}=0.20	0.5607	0.3222	0.3060	0.2961	0.3041	0.3306	0.3161	0.3108	0.2894
b_{73}=0.30	0.4382	0.3161	0.3099	0.3076	0.2956	0.3249	0.3382	0.3043	0.2919
b_{83}=0.40	0.2596	0.3159	0.2837	0.2659	0.2561	0.3147	0.3227	0.2594	0.2939
r_f=0.50	**0.0000**	0.0631	0.0126	0.1006	0.0115	0.0108	0.1882	0.0024	0.1789
λ_1=1.00	0.2584	0.4067	0.3165	0.3168	0.3005	0.3243	0.3799	0.3024	0.3679
λ_2=2.00	0.5779	0.5058	0.3859	0.6547	0.4073	0.6199	0.6605	0.7137	0.5515
λ_3=0.00	0.4461	0.5581	0.4793	0.4686	0.4660	0.4882	0.4859	0.4576	0.4840
ϕ_{21}=0.00	0.3044	0.3734	0.3446	0.3355	0.3397	0.3393	0.3454	0.3481	0.3749
ϕ_{31}=0.00	0.4530	0.3691	0.3566	0.3610	0.3532	0.3720	0.3375	0.3447	0.4105
ϕ_{32}=0.00	0.1818	0.2706	0.2210	0.2327	0.2074	0.2373	0.2437	0.2161	0.3363

Remarks: The values in boldface are fixed

Table C.5: Simulation result - RMSE

Design E	Factor loadings = B2			Factor correlation = $\Phi 1$			$\lambda_1 = 1$ $\lambda_2 = 2$ $\lambda_3 = 3$		
Constraints	Exact	1a	1b	2a	2b	2c	3a	3b	3aR
				$n = 2000$					
No of cases converaged	96	87	100	94	100	89	89	93	94
True value of parameters									
b_{11}=0.30	0.0578	0.0645	0.0611	0.0606	0.0573	0.0590	0.0670	0.0596	0.0586
b_{21}=0.35	0.0441	0.0499	0.0474	0.0473	0.0440	0.0487	0.0517	0.0456	0.0429
b_{31}=0.40	0.0485	0.0520	0.0508	0.0509	0.0479	0.0508	0.0550	0.0488	0.0462
b_{41}=0.45	0.0539	0.0604	0.0572	0.0560	0.0531	0.0569	0.0631	0.0553	0.0517
b_{51}=0.50	0.0718	0.0963	0.0882	0.0865	0.0733	0.0979	0.1011	0.0762	0.0944
b_{61}=0.55	0.0245	0.0415	0.0354	0.0323	0.0266	0.0416	0.0434	0.0289	0.0414
b_{71}=0.60	0.0329	0.0505	0.0441	0.0423	0.0346	0.0503	0.0530	0.0356	0.0495
b_{81}=0.65	**0.0000**	0.0442	0.0326	0.0269	0.0087	0.0446	0.0461	0.0154	0.0437
b_{12}=1.00	**0.0000**	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0281
b_{22}=0.60	0.0198	0.0212	0.0198	0.0208	0.0195	0.0197	0.0208	0.0197	0.0227
b_{32}=0.70	0.0206	0.0208	0.0208	0.0201	0.0205	0.0204	0.0222	0.0210	0.0233
b_{42}=0.80	0.0216	0.0237	0.0220	0.0233	0.0215	0.0221	0.0260	0.0214	0.0261
b_{53}=1.00	**0.0000**	0.0335	0.0244	0.0200	0.0064	0.0346	0.0372	0.0113	0.0404
b_{63}=0.20	0.0307	0.0304	0.0296	0.0305	0.0302	0.0306	0.0317	0.0310	0.0300
b_{73}=0.30	0.0324	0.0314	0.0312	0.0319	0.0321	0.0321	0.0332	0.0324	0.0317
b_{83}=0.40	0.0318	0.0297	0.0302	0.0310	0.0315	0.0318	0.0320	0.0320	0.0305
r_f=0.50	**0.0000**	0.1184	0.0364	0.1411	0.0380	0.0323	0.1625	0.0088	0.1658
λ_1=1.00	0.1643	0.2450	0.1741	0.2745	0.1750	0.1791	0.3199	0.1697	0.3160
λ_2=2.00	0.0962	0.1264	0.1002	0.1307	0.0982	0.1003	0.1416	0.0985	0.1391
λ_3=3.00	0.0869	0.2080	0.1741	0.1636	0.1054	0.2122	0.2224	0.1204	0.2366
ϕ_{21}=0.50	0.0794	0.0980	0.0895	0.0877	0.0796	0.0949	0.1016	0.0821	0.0804
ϕ_{31}=0.50	0.1279	0.1907	0.1697	0.1623	0.1320	0.1995	0.2047	0.1411	0.2002
ϕ_{32}=0.00	0.0407	0.0412	0.0399	0.0394	0.0400	0.0382	0.0416	0.0394	0.0496
				$n = 200$					
No of cases converaged	94	97	95	94	95	94	99	91	98
True values of parameters									
b_{11}=0.30	0.2044	0.2210	0.2007	0.2005	0.2039	0.2207	0.2254	0.2063	0.2307
b_{21}=0.35	0.1554	0.1776	0.1577	0.1570	0.1581	0.1776	0.1856	0.1613	0.1682
b_{31}=0.40	0.1734	0.1959	0.1734	0.1721	0.1734	0.1954	0.2063	0.1749	0.1839
b_{41}=0.45	0.1947	0.2240	0.1973	0.1988	0.1963	0.2207	0.2386	0.1992	0.2250
b_{51}=0.50	0.2154	0.2418	0.2186	0.3173	0.2179	0.2416	0.4535	0.2164	0.3162
b_{61}=0.55	0.0934	0.1241	0.0966	0.1007	0.0977	0.1182	0.1466	0.0971	0.1448
b_{71}=0.60	0.0987	0.1313	0.1055	0.1013	0.1023	0.1256	0.1580	0.0989	0.1618
b_{81}=0.65	**0.0000**	0.1036	0.0315	0.0200	0.0040	0.1016	0.1452	0.0077	0.1474
b_{12}=1.00	**0.0000**	0.0399	0.0148	0.0111	0.0028	0.0408	0.0470	0.0052	0.1066
b_{22}=0.60	0.0625	0.0727	0.0634	0.0660	0.0637	0.0730	0.0811	0.0631	0.0909
b_{32}=0.70	0.0695	0.0825	0.0703	0.0833	0.0705	0.0816	0.0954	0.0697	0.1028
b_{42}=0.80	0.0693	0.0805	0.0700	0.0752	0.0694	0.0806	0.0878	0.0683	0.0981
b_{53}=1.00	**0.0000**	0.1076	0.0293	0.0209	0.0033	0.1101	0.1501	0.0067	0.1710
b_{63}=0.20	0.1066	0.1615	0.1098	0.1126	0.1126	0.1654	0.1822	0.1079	0.1759
b_{73}=0.30	0.1072	0.1463	0.1089	0.1141	0.1081	0.1515	0.1823	0.1029	0.1696
b_{83}=0.40	0.0996	0.1561	0.1003	0.1067	0.1019	0.1634	0.1965	0.0981	0.1807
r_f=0.50	**0.0000**	0.1121	0.0181	0.1605	0.0183	0.0161	0.2981	0.0037	0.2915
λ_1=1.00	0.8856	0.8427	0.8737	0.9111	0.8948	1.1488	0.9949	0.7439	0.9881
λ_2=2.00	0.4161	0.4310	0.4120	0.4327	0.4191	0.4487	0.4779	0.3633	0.4744
λ_3=3.00	0.5853	0.5848	0.5909	0.6823	0.5890	0.8389	0.9443	0.4718	0.8424
ϕ_{21}=0.50	0.2725	0.3439	0.2749	0.2823	0.2749	0.3492	0.4647	0.2787	0.5198
ϕ_{31}=0.50	0.3730	0.5088	0.4043	0.5223	0.3805	0.5005	0.7509	0.3749	0.7328
ϕ_{32}=0.00	0.1832	0.1664	0.1731	0.2002	0.1819	0.1833	0.1829	0.1772	0.2604

Remarks: The values in boldface are fixed

Table C.6: Simulation result - RMSE

Design F	Factor loadings = B2			Factor correlation = $\Phi 1$			$\lambda_1 = 1$ $\lambda_2 = 2$ $\lambda_3 = 0$		
Constraints	Exact	1a	1b	2a	2b	2c	3a	3b	3aR
				$n = 2000$					
No of cases converaged	100	100	100	100	100	100	100	100	100
True value of parameters									
b_{11}=0.30	0.0673	0.1259	0.0752	0.0758	0.0689	0.0968	0.1968	0.0703	0.1792
b_{21}=0.35	0.0450	0.1184	0.0883	0.0916	0.0560	0.1397	0.1725	0.0515	0.1506
b_{31}=0.40	0.0497	0.1437	0.1301	0.1317	0.0808	0.1765	0.1991	0.0666	0.1790
b_{41}=0.45	0.0514	0.1523	0.1141	0.1176	0.0690	0.1736	0.2163	0.0669	0.1964
b_{51}=0.50	0.1350	0.1573	0.1345	0.1411	0.1325	0.1479	0.1736	0.1393	0.2009
b_{61}=0.55	0.0420	0.0834	0.0511	0.0523	0.0393	0.0795	0.1076	0.0443	0.1111
b_{71}=0.60	0.0356	0.0785	0.0424	0.0387	0.0293	0.0752	0.1039	0.0374	0.1063
b_{81}=0.65	**0.0000**	0.0739	0.0311	0.0231	0.0049	0.0740	0.0984	0.0092	0.0995
b_{12}=1.00	**0.0000**	0.0038	0.0000	0.0000	0.0000	0.0005	0.0105	0.0000	0.0625
b_{22}=0.60	0.0397	0.1316	0.1019	0.1086	0.0692	0.1620	0.1903	0.0478	0.1626
b_{32}=0.70	0.0191	0.1615	0.1461	0.1513	0.0988	0.2000	0.2211	0.0682	0.1971
b_{42}=0.80	0.0480	0.1733	0.1331	0.1412	0.0868	0.2021	0.2412	0.0662	0.2471
b_{53}=1.00	**0.0000**	0.0581	0.0348	0.0283	0.0072	0.0604	0.0807	0.0109	0.0917
b_{63}=0.20	0.1022	0.1534	0.1199	0.1228	0.1052	0.1506	0.1800	0.1063	0.1858
b_{73}=0.30	0.1018	0.1573	0.1220	0.1234	0.1068	0.1554	0.1860	0.1057	0.1894
b_{83}=0.40	0.0908	0.1604	0.1215	0.1225	0.1032	0.1608	0.1897	0.0958	0.1903
r_f=0.50	**0.0000**	0.1099	0.0446	0.1205	0.0459	0.0426	0.1411	0.0118	0.1413
λ_1=1.00	0.1631	0.3458	0.2453	0.3184	0.2283	0.3000	0.4295	0.1712	0.4185
λ_2=2.00	0.0570	0.2141	0.0675	0.1097	0.0600	0.1275	0.3881	0.0624	0.3305
λ_3=0.00	0.2159	0.2911	0.2568	0.2818	0.2539	0.2763	0.3083	0.2178	0.3466
ϕ_{21}=0.50	0.1422	0.2081	0.1743	0.1716	0.1703	0.2034	0.2401	0.1353	0.1877
ϕ_{31}=0.50	0.2613	0.3055	0.2803	0.2928	0.2643	0.2988	0.3266	0.2727	0.3414
ϕ_{32}=0.00	0.0681	0.0904	0.0750	0.0825	0.0876	0.0842	0.0934	0.0632	0.1121
				$n = 200$					
No of cases converaged	95	95	94	94	95	96	95	96	96
True values of parameters									
b_{11}=0.30	0.2786	0.2573	0.2720	0.2428	0.2844	0.2821	0.2740	0.2846	0.2869
b_{21}=0.35	0.3256	0.2897	0.3172	0.2973	0.3257	0.2850	0.2772	0.3310	0.2796
b_{31}=0.40	0.2480	0.3229	0.2353	0.2167	0.2490	0.2657	0.3359	0.2510	0.2856
b_{41}=0.45	0.3094	0.2909	0.3008	0.2927	0.3058	0.3268	0.3371	0.3087	0.3346
b_{51}=0.50	0.2450	0.2443	0.2433	0.2833	0.2464	0.3189	0.3629	0.2465	0.3558
b_{61}=0.55	0.0896	0.1149	0.0852	0.0973	0.0893	0.1091	0.1446	0.0894	0.1495
b_{71}=0.60	0.0987	0.1365	0.0976	0.1188	0.0983	0.1312	0.1521	0.0984	0.1568
b_{81}=0.65	**0.0000**	0.0801	0.0163	0.0112	0.0019	0.0693	0.1295	0.0037	0.1374
b_{12}=1.00	**0.0000**	0.0307	0.0121	0.0079	0.0018	0.0367	0.0451	0.0033	0.1021
b_{22}=0.60	0.2912	0.2845	0.2672	0.2448	0.2911	0.2697	0.2664	0.2972	0.2725
b_{32}=0.70	0.2137	0.3234	0.1913	0.1753	0.2141	0.2361	0.3157	0.2181	0.2811
b_{42}=0.80	0.2459	0.2331	0.2301	0.2258	0.2420	0.2688	0.3010	0.2472	0.2964
b_{53}=1.00	**0.0000**	0.0952	0.0344	0.0213	0.0049	0.1034	0.1361	0.0093	0.1637
b_{63}=0.20	0.1876	0.1921	0.1891	0.1820	0.1878	0.2159	0.2204	0.1903	0.2201
b_{73}=0.30	0.1989	0.2273	0.2004	0.2161	0.1995	0.2476	0.2488	0.2004	0.2461
b_{83}=0.40	0.1830	0.1863	0.1823	0.1765	0.1838	0.2201	0.2210	0.1853	0.2180
r_f=0.50	**0.0000**	0.1467	0.0230	0.1767	0.0228	0.0202	0.2539	0.0047	0.2468
λ_1=1.00	0.3892	0.4822	0.4035	0.4460	0.3850	0.4415	0.6515	0.4102	0.6128
λ_2=2.00	0.4446	0.3998	0.4302	0.3653	0.4563	0.4557	0.4645	0.4617	0.4676
λ_3=0.00	0.4634	0.4717	0.4769	0.4889	0.4613	0.4976	0.4805	0.4859	0.4900
ϕ_{21}=0.50	0.3868	0.4499	0.4040	0.3687	0.3904	0.4354	0.5521	0.3906	0.6129
ϕ_{31}=0.50	0.4897	0.5275	0.4689	0.5204	0.4929	0.5222	0.6678	0.4954	0.7451
ϕ_{32}=0.00	0.2246	0.2264	0.2426	0.2329	0.2236	0.2415	0.2135	0.2283	0.2563

Remarks: The values in boldface are fixed

Table C.7: Simulation result - RMSE

Design G	Factor loadings = B2		Factor correlation = $\Phi 2$			$\lambda_1 = 1$ $\lambda_2 = 2$ $\lambda_3 = 3$			
Constraints	Exact	1a	1b	2a	2b	2c	3a	3b	3aR
$n = 2000$									
No of cases converaged	100	99	99	100	99	100	96	100	96
True value of parameters									
b_{11}=0.30	0.1681	0.1486	0.1379	0.1556	0.1362	0.1462	0.1472	0.1631	0.1954
b_{21}=0.35	0.1017	0.0918	0.0860	0.0937	0.0869	0.0888	0.0902	0.0995	0.1240
b_{31}=0.40	0.1186	0.1045	0.1018	0.1072	0.1000	0.1029	0.1037	0.1156	0.1382
b_{41}=0.45	0.1334	0.1242	0.1160	0.1293	0.1138	0.1222	0.1241	0.1305	0.1616
b_{51}=0.50	0.0768	0.1343	0.0959	0.0928	0.0792	0.1340	0.1508	0.0809	0.1739
b_{61}=0.55	0.0307	0.0363	0.0337	0.0327	0.0315	0.0362	0.0366	0.0317	0.0414
b_{71}=0.60	0.0293	0.0409	0.0339	0.0329	0.0301	0.0406	0.0459	0.0305	0.0499
b_{81}=0.65	**0.0000**	0.0407	0.0200	0.0168	0.0027	0.0407	0.0466	0.0056	0.0593
b_{12}=1.00	**0.0000**	0.0173	0.0158	0.0152	0.0102	0.0188	0.0198	0.0126	0.0331
b_{22}=0.60	0.0176	0.0188	0.0189	0.0188	0.0183	0.0189	0.0192	0.0188	0.0281
b_{32}=0.70	0.0171	0.0193	0.0189	0.0193	0.0181	0.0197	0.0200	0.0186	0.0276
b_{42}=0.80	0.0187	0.0227	0.0216	0.0232	0.0206	0.0220	0.0240	0.0211	0.0321
b_{53}=1.00	**0.0000**	0.0685	0.0417	0.0392	0.0094	0.0645	0.0707	0.0159	0.0686
b_{63}=0.20	0.0312	0.0334	0.0320	0.0320	0.0310	0.0328	0.0338	0.0310	0.0334
b_{73}=0.30	0.0325	0.0358	0.0339	0.0334	0.0320	0.0356	0.0358	0.0322	0.0348
b_{83}=0.40	0.0350	0.0394	0.0368	0.0369	0.0348	0.0393	0.0396	0.0347	0.0380
r_f=0.50	**0.0000**	0.1294	0.0377	0.1446	0.0383	0.0348	0.1863	0.0088	0.1861
λ_1=1.00	0.1941	0.2883	0.1989	0.3116	0.2016	0.1943	0.3626	0.1897	0.3628
λ_2=2.00	0.1753	0.1732	0.1301	0.1941	0.1291	0.1604	0.2050	0.1670	0.2175
λ_3=3.00	0.1184	0.2702	0.1890	0.1762	0.1260	0.2736	0.2698	0.1317	0.3640
ϕ_{21}=0.00	0.1542	0.1491	0.1543	0.1526	0.1530	0.1445	0.1468	0.1530	0.1865
ϕ_{31}=0.00	0.1037	0.1598	0.1422	0.1341	0.1084	0.1602	0.1656	0.1131	0.1912
ϕ_{32}=0.00	0.0484	0.0595	0.0485	0.0524	0.0416	0.0723	0.0653	0.0529	0.0856
$n = 200$									
No of cases converaged	99	97	99	99	99	97	97	96	96
True values of parameters									
b_{11}=0.30	0.2552	0.2516	0.2455	0.3421	0.2543	0.3958	0.2700	0.2571	0.3023
b_{21}=0.35	0.1812	0.1855	0.1790	0.2305	0.1807	0.2720	0.1992	0.1834	0.2175
b_{31}=0.40	0.2051	0.2052	0.2003	0.2489	0.2042	0.2970	0.2137	0.2060	0.2289
b_{41}=0.45	0.2391	0.2238	0.2250	0.2690	0.2382	0.3223	0.2381	0.2405	0.2631
b_{51}=0.50	0.2627	0.3801	0.2852	0.2908	0.2632	0.3534	0.3989	0.2781	0.3750
b_{61}=0.55	0.1088	0.1251	0.1122	0.1084	0.1090	0.1287	0.1349	0.1125	0.1470
b_{71}=0.60	0.1135	0.1256	0.1137	0.1189	0.1140	0.1349	0.1499	0.1151	0.1516
b_{81}=0.65	**0.0000**	0.0680	0.0170	0.0121	0.0020	0.0748	0.1124	0.0041	0.1286
b_{12}=1.00	**0.0000**	0.0560	0.0357	0.0293	0.0075	0.0576	0.0617	0.0128	0.0973
b_{22}=0.60	0.0629	0.0635	0.0632	0.0630	0.0627	0.0625	0.0643	0.0632	0.0778
b_{32}=0.70	0.0666	0.0671	0.0657	0.0641	0.0661	0.0662	0.0690	0.0665	0.0804
b_{42}=0.80	0.0716	0.0717	0.0712	0.0714	0.0715	0.0715	0.0733	0.0726	0.0880
b_{53}=1.00	**0.0000**	0.1312	0.0365	0.0259	0.0043	0.1377	0.1843	0.0085	0.1912
b_{63}=0.20	0.1365	0.1590	0.1494	0.1426	0.1367	0.1637	0.1619	0.1405	0.1637
b_{73}=0.30	0.1319	0.1637	0.1503	0.1462	0.1322	0.1695	0.1687	0.1367	0.1685
b_{83}=0.40	0.1254	0.1679	0.1438	0.1412	0.1256	0.1721	0.1701	0.1300	0.1737
r_f=0.50	**0.0000**	0.0758	0.0140	0.1110	0.0140	0.0124	0.1978	0.0029	0.1952
λ_1=1.00	0.7968	0.7887	0.8073	0.8166	0.7941	0.7648	0.8125	0.8069	0.8187
λ_2=2.00	0.3839	0.3915	0.3966	0.5811	0.3815	0.4703	0.3942	0.3958	0.4859
λ_3=3.00	0.5542	0.8610	0.5796	0.5485	0.5550	0.8257	0.8820	0.5764	0.9898
ϕ_{21}=0.00	0.3175	0.3062	0.3123	0.2886	0.3169	0.3022	0.3317	0.3235	0.4235
ϕ_{31}=0.00	0.3668	0.4305	0.3746	0.3814	0.3664	0.3825	0.4416	0.3921	0.5429
ϕ_{32}=0.00	0.1818	0.1966	0.1891	0.1896	0.1815	0.2235	0.2432	0.1901	0.1975

Remarks: The values in boldface are fixed

Table C.8: Simulation result - RMSE

Design H	Factor loadings = B2			Factor correlation = $\Phi 2$			$\lambda_1 = 1$ $\lambda_2 = 2$ $\lambda_3 = 0$		
Constraints	Exact	1a	1b	2a	2b	2c	3a	3b	3aR
				$n = 2000$					
No of cases converaged	100	99	98	100	100	100	100	100	99
True value of parameters									
b_{11}=0.30	0.1486	0.1667	0.1297	0.1627	0.1214	0.1617	0.1605	0.1513	0.2151
b_{21}=0.35	0.0937	0.1139	0.0873	0.1098	0.0766	0.1163	0.1210	0.0974	0.1478
b_{31}=0.40	0.1096	0.1328	0.0995	0.1297	0.0879	0.1371	0.1423	0.1147	0.1693
b_{41}=0.45	0.1280	0.1542	0.1169	0.1510	0.1022	0.1590	0.1662	0.1320	0.1971
b_{51}=0.50	0.0994	0.1235	0.1145	0.1117	0.1003	0.1226	0.1263	0.1022	0.1352
b_{61}=0.55	0.0325	0.0305	0.0313	0.0326	0.0321	0.0331	0.0299	0.0321	0.0358
b_{71}=0.60	0.0196	0.0322	0.0214	0.0202	0.0194	0.0319	0.0338	0.0196	0.0467
b_{81}=0.65	**0.0000**	0.0391	0.0165	0.0126	0.0024	0.0369	0.0418	0.0048	0.0611
b_{12}=1.00	**0.0000**	0.0309	0.0201	0.0203	0.0112	0.0412	0.0638	0.0140	0.0668
b_{22}=0.60	0.0343	0.0563	0.0562	0.0598	0.0351	0.0742	0.0958	0.0358	0.0899
b_{32}=0.70	0.0381	0.0626	0.0619	0.0679	0.0391	0.0846	0.1132	0.0410	0.1040
b_{42}=0.80	0.0444	0.0762	0.0740	0.0818	0.0458	0.0994	0.1412	0.0473	0.1231
b_{53}=1.00	**0.0000**	0.0791	0.0390	0.0300	0.0065	0.0808	0.1060	0.0125	0.1081
b_{63}=0.20	0.1084	0.1211	0.1262	0.1277	0.1089	0.1309	0.1326	0.1121	0.1313
b_{73}=0.30	0.1142	0.1320	0.1339	0.1362	0.1154	0.1413	0.1445	0.1192	0.1459
b_{83}=0.40	0.1242	0.1474	0.1462	0.1493	0.1257	0.1566	0.1611	0.1296	0.1623
r_f=0.50	**0.0000**	0.1072	0.0403	0.1173	0.0421	0.0377	0.1438	0.0108	0.1443
λ_1=1.00	0.1179	0.2736	0.2017	0.2733	0.1340	0.2582	0.3960	0.1248	0.3615
λ_2=2.00	0.1763	0.2539	0.1966	0.2530	0.1369	0.2617	0.3143	0.1849	0.3481
λ_3=0.00	0.1513	0.2544	0.2189	0.2263	0.1515	0.2764	0.3556	0.1584	0.3165
ϕ_{21}=0.00	0.1348	0.1405	0.1382	0.1366	0.1348	0.1494	0.1729	0.1349	0.1991
ϕ_{31}=0.00	0.2144	0.2430	0.2394	0.2375	0.2164	0.2435	0.2368	0.2209	0.2439
ϕ_{32}=0.00	0.0769	0.0896	0.0737	0.0842	0.0715	0.0907	0.1095	0.0835	0.0823
				$n = 200$					
No of cases converaged	96	93	92	94	93	90	91	96	91
True values of parameters									
b_{11}=0.30	0.2753	0.2629	0.2836	0.3964	0.2895	0.4691	0.2582	0.2808	0.2924
b_{21}=0.35	0.1980	0.1856	0.2072	0.2577	0.2103	0.3008	0.1893	0.2060	0.3834
b_{31}=0.40	0.2341	0.2385	0.2344	0.2968	0.2340	0.3152	0.2074	0.2474	0.4703
b_{41}=0.45	0.3368	0.2227	0.2464	0.2940	0.2355	0.3341	0.2230	0.2710	0.3414
b_{51}=0.50	0.2858	0.4513	0.4176	0.3524	0.3686	0.3539	0.3496	0.3222	0.3582
b_{61}=0.55	0.1317	0.1415	0.1348	0.0786	0.1665	0.0876	0.1058	0.1287	0.3508
b_{71}=0.60	0.2787	0.2545	0.2647	0.1204	0.3131	0.2128	0.2758	0.2895	0.3826
b_{81}=0.65	**0.0000**	0.0440	0.0151	0.0121	0.0023	0.0550	0.0781	0.0044	0.0839
b_{12}=1.00	**0.0000**	0.0635	0.0373	0.0304	0.0077	0.0642	0.0762	0.0132	0.0940
b_{22}=0.60	0.1537	0.1309	0.1484	0.1325	0.1547	0.2005	0.0987	0.1545	0.3770
b_{32}=0.70	0.1752	0.1575	0.1636	0.1416	0.1728	0.1801	0.1215	0.1864	0.4619
b_{42}=0.80	0.3109	0.1738	0.1935	0.1778	0.1737	0.2376	0.1433	0.2265	0.3228
b_{53}=1.00	**0.0000**	0.0902	0.0295	0.0212	0.0041	0.1154	0.1515	0.0079	0.1547
b_{63}=0.20	0.2558	0.2829	0.2744	0.2467	0.2776	0.2462	0.2430	0.2540	0.4436
b_{73}=0.30	0.3903	0.4134	0.3943	0.3046	0.4111	0.3599	0.4103	0.3883	0.5018
b_{83}=0.40	0.2690	0.3295	0.3040	0.3081	0.2706	0.3106	0.3054	0.2687	0.3154
r_f=0.50	**0.0000**	0.0965	0.0185	0.1410	0.0183	0.0157	0.2076	0.0038	0.2134
λ_1=1.00	0.4518	0.5022	0.4577	0.4799	0.4408	0.5273	0.5417	0.7024	0.5639
λ_2=2.00	0.5093	0.4248	0.4573	0.7223	0.5133	0.8634	0.3237	0.4937	0.3554
λ_3=0.00	0.4989	0.5280	0.5365	0.4702	0.5334	0.5542	0.5091	0.7036	0.5378
ϕ_{21}=0.00	0.2687	0.3361	0.2813	0.2695	0.2578	0.3523	0.3475	0.3961	0.3894
ϕ_{31}=0.00	0.4667	0.3998	0.4301	0.3892	0.4496	0.4083	0.3866	0.4335	0.4288
ϕ_{32}=0.00	0.2028	0.2536	0.2186	0.2110	0.1860	0.3069	0.2788	0.3014	0.2575

Remarks: The values in boldface are fixed

Appendix D

Mx input script

D.1 Stochastic constraints Case 1

! This is a Mx sample input script for a simulation design A.

! Stochastic constraints on r_f and reference factor loadings.

! Case 1: Gamma is diagonal matrix with diagonal elements all equal to sigma.

! Start at the estimates obtained from the model with exact constraints.

groups= 2

♯define factors 3

♯define vars 8

♯define ncon 4

G1

Data NI = 8 NO = 2000

Rectangular C: simcase1.raw

Label r1 r2 r3 r4 r5 r6 r7 r8

Begin Matrices;

B Full vars factors free

P Stan factors factors free

E Diag vars vars free

H Full vars 1 free

M Full factors 1 free

End Matrices;

Fix B 5 2 B 6 2 B 7 2 B 8 2 B 1 3 B 2 3 B 3 3 B 4 3

Value 0.0 B 5 2 B 6 2 B 7 2 B 8 2 B 1 3 B 2 3 B 3 3 B 4 3

Start 0.65 B 1 1 1

Start 0.71 B 1 2 1

Start 0.75 B 1 3 1

Start 0.80 B 1 4 1

Start 0.84 B 1 5 1

Start 0.89 B 1 6 1

Start 0.95 B 1 7 1

Start 1.00 B 1 8 1

Start 1.00 B 1 1 2

Start 0.59 B 1 2 2

Start 0.69 B 1 3 2

Start 0.79 B 1 4 2

Start 1.00 B 1 5 3

Start 0.20 B 1 6 3

Start 0.30 B 1 7 3

Start 0.40 B 1 8 3

Start 1.02 B 1 1 1

Start 2.00 B 1 2 1

Start 2.99 B 1 3 1

Start 0.0 P 1 2 1 P 1 3 1 P 1 3 2

Start 0.5 E 1 1 1 E 1 2 2 E 1 3 3 E 1 4 4 E 1 5 5 E 1 6 6 E 1 7 7 E 1 8 8

Mean $H + B \star M/$

Covariance $B \star P \star B' + E/$

End Group

G2 !Group 2 user-defined fit function

Data NI = 0

Begin Matrices;

Appendix D

Mx input script

D.1 Stochastic constraints Case 1

! This is a Mx sample input script for a simulation design A.

! Stochastic constraints on r_f and reference factor loadings.

! Case 1: Gamma is diagonal matrix with diagonal elements all equal to sigma.

! Start at the estimates obtained from the model with exact constraints.

groups= 2

♯define factors 3

♯define vars 8

♯define ncon 4

G1

Data NI = 8 NO = 2000

Rectangular C: simcase1.raw

Label r1 r2 r3 r4 r5 r6 r7 r8

Begin Matrices;

B Full vars factors free

P Stan factors factors free

E Diag vars vars free

H Full vars 1 free

M Full factors 1 free

End Matrices;

Fix B 5 2 B 6 2 B 7 2 B 8 2 B 1 3 B 2 3 B 3 3 B 4 3

Value 0.0 B 5 2 B 6 2 B 7 2 B 8 2 B 1 3 B 2 3 B 3 3 B 4 3

Start 0.65 B 1 1 1

Start 0.71 B 1 2 1

Start 0.75 B 1 3 1

Start 0.80 B 1 4 1

Start 0.84 B 1 5 1

Start 0.89 B 1 6 1

Start 0.95 B 1 7 1

Start 1.00 B 1 8 1

Start 1.00 B 1 1 2

Start 0.59 B 1 2 2

Start 0.69 B 1 3 2

Start 0.79 B 1 4 2

Start 1.00 B 1 5 3

Start 0.20 B 1 6 3

Start 0.30 B 1 7 3

Start 0.40 B 1 8 3

Start 1.02 B 1 1 1

Start 2.00 B 1 2 1

Start 2.99 B 1 3 1

Start 0.0 P 1 2 1 P 1 3 1 P 1 3 2

Start 0.5 E 1 1 1 E 1 2 2 E 1 3 3 E 1 4 4 E 1 5 5 E 1 6 6 E 1 7 7 E 1 8 8

Mean $H + B \star M/$

Covariance $B \star P \star B' + E/$

End Group

G2 !Group 2 user-defined fit function

Data NI = 0

Begin Matrices;

U Full ncon 1

G Full ncon 1 free

V Full 1 1

A Full 1 1

C Full 1 1

End Matrices;

Equate G 2 1 1 H 1 1 1 H 1 2 1 H 1 3 1 H 1 4 1 H 1 5 1 H 1 6 1 H 1 7 1 H 1 8 1

Equate G 2 2 1 B 1 1 2

Equate G 2 3 1 B 1 5 3

Equate G 2 4 1 B 1 8 1

Value 0.5 U 1 1

Value 1.0 U 2 1

Value 1.0 U 3 1

Value 1.0 U 4 1

Value 5 V 1 1

Value 0.1 A 1 1

Value 4.0 C 1 1

Start 0.5 G 2 1 1

Start 1.0 G 2 2 1

Start 1.0 G 2 3 1

Start 1.0 G 2 4 1

Compute $(V + C) \star (\backslash ln((U - G)' \star (U - G) + V \star A))/$

Option User-defined

End

D.2 Stochastic constraints Case 2

! This is a Mx sample input script for a simulation design A.

! Stochastic constraints on r_f and reference factor loadings.

! Case 2: Gamma is diagonal matrix with diagonal elements $sigma_j$.

! Start at the estimates obtained from the model with exact constraints.

groups= 2

♯define factors 3

♯define vars 8

♯define ncon 4

G1

Data NI = 8 NO = 2000

Rectangular C: simcase2.raw

Label r1 r2 r3 r4 r5 r6 r7 r8

Begin Matrices;

B Full vars factors free

P Stan factors factors free

E Diag vars vars free

H Full vars 1 free

M Full factors 1 free

End Matrices;

Fix B 5 2 B 6 2 B 7 2 B 8 2 B 1 3 B 2 3 B 3 3 B 4 3

Value 0.0 B 5 2 B 6 2 B 7 2 B 8 2 B 1 3 B 2 3 B 3 3 B 4 3

Start 0.65 B 1 1 1

Start 0.71 B 1 2 1

Start 0.75 B 1 3 1

Start 0.80 B 1 4 1

Start 0.84 B 1 5 1

Start 0.89 B 1 6 1

Start 0.95 B 1 7 1

Start 1.00 B 1 8 1

Start 1.00 B 1 1 2

Start 0.59 B 1 2 2

Start 0.69 B 1 3 2

Start 0.79 B 1 4 2

Start 1.00 B 1 5 3

Start 0.20 B 1 6 3

Start 0.30 B 1 7 3

Start 0.40 B 1 8 3

Start 1.02 B 1 1 1

Start 2.00 B 1 2 1

Start 2.99 B 1 3 1

Start 0.0 P 1 2 1 P 1 3 1 P 1 3 2

Start 0.5 E 1 1 1 E 1 2 2 E 1 3 3 E 1 4 4 E 1 5 5 E 1 6 6 E 1 7 7 E 1 8 8

Mean $H + B \star M/$

Covariance $B \star P \star B' + E/$

End Group

G2 ! Group 2

Data NI = 0

Begin Matrices;

U Full 1 1

W Full 1 1

X Full 1 1

Y Full 1 1

G Full 1 1 free

H Full 1 1 free

V Full 1 1

Q Full 1 1

B Full 1 1

D Full 1 1

C Full 1 1

E Full 1 1

J Full 1 1 free

K Full 1 1 free

End Matrices;

Equate G 2 1 1 H 1 1 1 H 1 2 1 H 1 3 1 H 1 4 1 H 1 5 1 H 1 6 1 H 1 7 1 H 1 8 1

Equate H 2 1 1 A 1 1 2

Equate J 2 1 1 A 1 5 3

Equate K 2 1 1 A 1 8 1

Value 0.5 U 1 1

Value 1.0 W 1 1

Value 1.0 X 1 1

Value 1.0 Y 1 1

Value 5 V 1 1

Value 5 Q 1 1

Value 0.01 B 1 1

Value 0.1 D 1 1

Value 1.0 C 1 1

Start 0.5 G 2 1 1

Start 1.0 H 2 1 1

Start 1.0 J 2 1 1

Start 1.0 K 2 1 1

Compute $(V + C) \star (\backslash ln((U - G)' \star (U - G) + V \star B)) + (Q + C) \star (\backslash ln((W - H)' \star (W - H) + Q \star D)) + (Q + C) \star (\backslash ln((X - J)' \star (X - J) + Q \star D)) + (Q + C) \star (\backslash ln((Y - K)' \star (Y - K) + Q \star D))/$

Option User-defined

End

D.3 Stochastic constraints Case 3

! This is a Mx sample input script for a simulation design A.

! Stochastic constraints on r_f and reference factor loadings.

! Case 3: Gamma is a general positive definite matrix.

! Start at the estimates obtained from the model with exact constraints.

```
groups= 2
♯define factors 3
♯define vars 8
♯define ncon 4
G1
Data NI = 8 NO = 2000
Rectangular C: simcase3.raw
Label r1 r2 r3 r4 r5 r6 r7 r8
Begin Matrices;
B Full vars factors free
P Stan factors factors free
E Diag vars vars free
H Full vars 1 free
M Full factors 1 free
End Matrices;
Fix B 5 2 B 6 2 B 7 2 B 8 2 B 1 3 B 2 3 B 3 3 B 4 3
Value 0.0 B 5 2 B 6 2 B 7 2 B 8 2 B 1 3 B 2 3 B 3 3 B 4 3
Start 0.65 B 1 1 1
Start 0.71 B 1 2 1
Start 0.75 B 1 3 1
Start 0.80 B 1 4 1
Start 0.84 B 1 5 1
Start 0.89 B 1 6 1
Start 0.95 B 1 7 1
```

```
Start 1.00 B 1 8 1
Start 1.00 B 1 1 2
Start 0.59 B 1 2 2
Start 0.69 B 1 3 2
Start 0.79 B 1 4 2
Start 1.00 B 1 5 3
Start 0.20 B 1 6 3
Start 0.30 B 1 7 3
Start 0.40 B 1 8 3
Start 1.02 B 1 1 1
Start 2.00 B 1 2 1
Start 2.99 B 1 3 1
Start 0.0 P 1 2 1 P 1 3 1 P 1 3 2
Start 0.5 E 1 1 1 E 1 2 2 E 1 3 3 E 1 4 4 E 1 5 5 E 1 6 6 E 1 7 7 E 1 8 8
Mean H + B * M/
Covariance B * P * B' + E/
End Group
G2 ! Group 2
Data NI = 0
Begin Matrices;
U Full ncon 1
G Full ncon 1 free
P Full 1 1 ! store the degree of freedom +1 (rho+1)
R Iden ncon ncon
End Matrices;
Equate G 2 1 1 H 1 1 1 H 1 2 1 H 1 3 1 H 1 4 1 H 1 5 1 H 1 6 1 H 1 7 1 H 1 8 1
Equate G 2 2 1 A 1 1 2
Equate G 2 3 1 A 1 5 3
Equate G 2 4 1 A 1 8 1
Value 0.5 U 1 1
```

```
Value 1.0 U 2 1
Value 1.0 U 3 1
Value 1.0 U 4 1
Value 6 P 1 1
Start 0.5 G 2 1 1
Start 1.0 G 2 2 1
Start 1.0 G 2 3 1
Start 1.0 G 2 4 1
Begin Algebra;
A = (U * G) * (U * G)';
C = A + R;
End Algebra;
Compute P * (\ln(\det(C))) /
Option User-defined
End
```

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